First approach to ∞-categories

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Abstract

This wants to be an extended version of my exam on the course of Category Theory by Filippo Bonchi.
It is a rewritten recap -in no way original- of different sources which I will now cite once and for all: Lurie's [5], Joyal's [4] and Cisinksi's [2]. As a must, I have to cite **nCatLab** [1] too.

Contents

Introduction

This document wants to be a brief recap on what I studied when learning ∞-categories. The main goal will be that of defining ∞-cats (I will sometimes call them like this) and hopefully some "out"sights[1] on the consequences the theory. In particular, we will state and give an outline of the proof of a result of Boardman and Vogt which tells us how (a category associated to) an ∞-cat keeps track of homotopical information.
Just to make things faster, I will give the main definition of this expository article now.

Definition 0.1. An ∞-category is a simplicial set X such that the inclusion-induced maps

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$$

[1] As of now I would not be able to give *in*sights.

are surjective for any $n \geq 2,\ 0 < k < n$.

The next sections will try to give precise definitions for all the ingredients used in the definitions. For the one of you which know the definition of *Kan complexes* (and almost certainly already encountered this definition of ∞-cats) the striking similarity appears. This "model" (in a meta-mathematical way) of ∞-cats is called under the name of *weak Kan complexes*.

1 Presheaves, Yoneda and Simplicial Sets

1.1 What is a presheaf and (yet another) Yoneda

Simplicial sets are fundamental objects in different areas of mathematics. In their down-to-Earth form they are simply objects which (in some sense) resemble the standard (geometrical) simplex, or the "simplest n-dimensional convex". With these intuition they serve as very "comfortable" space of parameters for very concrete things (such as optimization algorithm). They - however - are the very starting brick of a much greater skyscraper: *homotopy theory*, in its broadest sense. Topologically, they serve as foundational constructing blocks for well-behaved spaces and (sometimes, and hopefully so) simply studying how simplices can be attached to a space, and the way to deform these attachments gives us a nice ID of the space. Surely you won't know about some work-off-the-books just by looking at someone's ID, but in the vast majority of the time, you really don't need these kinds of information. The same happens to topological spaces.
Simplices are nice combinatorial and algebraic objects: they are *naturally* totally ordered finite sets and studying maps from and to these objects can reveal important combinatorial information about other sets.
Some algebraic conditions on polynomial rings may be encoded in simplicial sets (informally, unions of simplices) and algebraic properties may be obtained by doing some operations on associated simplicial sets (one of those very nice property is "shellability").
What we said so far is, in my opinion, a simple reason on **why** to study simplicial sets: they serve as a natural way to generalize already working and present ideas on more abstract contexts.
Let us start with a bit of formalism.
The following definition is not necessary for the rest of the exposition, but I think it simplifies some concepts and is overall a very common notion.

Definition 1.1. A **presheaf** over a category $\mathcal{A}$ is a functor

$$X : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}.$$

Given a $a \in Ob(\mathcal{A})$, the set $X(a)$ (or X_a or simply Xa) is called *fibre* over a and its elements are the *sections* over a.
Given the definition, we can naturally call *morphism of presheves* a natural transformation between a presheaf X and a presheaf Y.
Omitting the trivial checks, one convinces himself that sheaves over a category $\mathcal{A}$ and morphism between sheaves forms a category. We will call this category $\hat{\mathcal{A}}$.
Continuing this excursus on presheaves, let me give a fundamental result in category

theory, namely the (or *a*) **Yoneda Lemma**.
In the following section $\mathcal{A}$ will always be locally small (unless otherwise stated).

Definition 1.2. The *Yoneda embedding* is the morphism

$$\mathcal{A} \to \hat{\mathcal{A}}$$

sending $a \mapsto h_a$ where $h_a = \mathrm{Hom}_{\mathcal{A}}(\bullet, a)$.

Theorem 1.3 (Yoneda Lemma)**.** *Given X an object of $\hat{\mathcal{A}}$ there is a natural bijection*

$$Hom_{\hat{\mathcal{A}}}(h_a, X) \xrightarrow{\cong} X_a$$

given by

$$(u : h_a \to X) \mapsto u_a(1_a).$$

Proof. This fundamental result has a rather simple proof.
We have to show an inverse. Given $s \in X_a$ a section, we associate to a morphism $(u : c \to a)$ an element of X_c simply choosing $u^*(s)$.
This is *clearly* an inverse. □

Using this we obtain that the Yoneda embedding is a fully faithful functor: clearly $\mathrm{Hom}_{\hat{\mathcal{A}}}(h_x, h_y) \cong \mathrm{Hom}_{\mathcal{A}}(x, y)$.
Let us proceed on the line of Cisinski [2], by introducing the useful notion of Grothendieck construction of a presheaf X.

Definition 1.4. Given X a presheaf over $\mathcal{A}$ we call *Groethendieck construction of* X (or also *category of elements*) the category whose objects are pairs (a, s) with $a \in \mathcal{A}, s \in X_a$ and a morphism $u : (a, s) \to (b, t)$ for each morphism $u : a \to b$ in A such that $u^*(t) = s$.

We will indicate this category with $\mathcal{A}/X$ and notice that we have a faithful functor

$$\varphi_X : \mathcal{A}/X \to \hat{\mathcal{A}}$$

sending $(a, s) \mapsto h_a$ and sending morphism u to itself (notice that $u : a \to b$ induces $h_a \to h_b$).
By using the Yoneda lemma one gets that the category of elements of X is precisely the category of maps $(h_a \xrightarrow{s} X)$ and the morphisms are commutative [2] triangles such as

$$\begin{array}{ccc} h_a & \xrightarrow{\quad u \quad} & h_b \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle t} & \\ & X & \end{array}$$

The following elementary fact, which I will not prove, is a variation of the Yoneda lemma.

Proposition 1.5. *The maps $h_a \to X$ exhibit X as a colimit under the functor φ_X.*

Let's end this "Yoneda excursus" by proving the following theorem, due to **Kan**.

[2] Reminder: this is the **real** commutativity.

Theorem 1.6 (Kan). *Let $\mathcal{A}$ be a* ***small*** *category and $\mathcal{C}$ a* ***locally small*** *category with all small colimits.*
For any functor $u : \mathcal{A} \to \mathcal{C}$ there exists a functor of evaluation at u defined via

$$u^* : \mathcal{C} \to \hat{\mathcal{A}}, \ \ \mathcal{C} \ni Y \mapsto u^*(Y) = (a \mapsto Hom_{\mathcal{C}}(u(a), Y)).$$

This functor has a left adjoint $u_! : \hat{\mathcal{A}} \to \mathcal{C}$.
Moreover there are unique natural isomorphisms $u(a) \simeq u_!(h_a)$ such that the induced bijection

$$Hom_{\mathcal{C}}(u_!(h_a), Y) \simeq Hom_{\mathcal{C}}(u(a), Y)$$

is the inverse of the Yoneda bijection

$$Hom_{\mathcal{C}}(u(a), Y) = u^*(Y)_a \simeq Hom_{\hat{\mathcal{A}}}(h_a, u^*(Y))$$

with the adjunction

$$Hom_{\hat{\mathcal{A}}}(h_a, u^*(Y)) \simeq Hom_{\mathcal{C}}(u_!(h_a), Y).$$

Proof. We start by proving that u^* has a left adjoint.
Given a presheaf X over A, take a colimit of the maps

$$\mathcal{A}/X \to \mathcal{C}, \ \ (a, s) \mapsto u(a)$$

and call it $u_!(X)$. If we choose the Yoneda embedding of $\mathcal{A}$ as presheaf, namely put $X = h_a$ we obtain a canonical isomorphism $u_!(h_a) \simeq u(a)$ because clearly $(a, 1_a)$ is a final object in the category of elements $\mathcal{A}/h_a$.
Now in full generality, taking a presheaf X we have the following identifications (I won't check naturality but it follows from naturality of Yoneda):

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(u_!(X), Y) &\simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim_{(a,s)} u(a), Y) && (1)\\
&\simeq \varprojlim_{(a,s)} \mathrm{Hom}_{\mathcal{C}}(u(a), Y) && (2)\\
&\simeq \varprojlim_{(a,s)} \mathrm{Hom}_{\hat{\mathcal{A}}}(h_a, u^*(Y)) \text{ by Yoneda} && (3)\\
&\simeq \mathrm{Hom}_{\hat{\mathcal{A}}}(\varinjlim_{(a,s)} h_a, u^*(Y)) && (4)\\
&\simeq \mathrm{Hom}_{\hat{\mathcal{A}}}(X, u^*(Y)) \text{ by proposition 1.5.} && (5)
\end{aligned}$$

All being natural in X, Y, these isomorphisms give us that $u_!$ is a left adjoint for u^*.
□

What one discovers is that every colimit preserving functor $\hat{\mathcal{A}} \to \mathcal{C}$ is of the form $u_!$ for an appropriato functor u: in particular, by the preceding theorem, every such functor has a right adjoint.

1.2 Simplicial sets...

While in my bachelor thesis the first chapter was an introduction to simplicial sets from a more mathematical point of view, here I am going to introduce them in a purely categorical way.

I will hopefully be able to give you the topological meaning of all this abstract formalism, the way I now[3] perceive them.
The main object of this section will be the category of finite linear order $\mathbf{\Delta}$.
Its objects are the totally ordered sets $[n]_{n\geq 0} = \{0 \leq 1 \leq 2 \leq \cdots \leq n\}$ and the morphisms are (non-strictly) increasing maps $[n] \to [m]$.

Definition 1.7. We call **sSet** the category of presheaves over $\mathbf{\Delta}$. The objects of **sSet** are called *simplicial sets*.

We will call the standard *n-simplex* Δ^n the functor $h_{[n]}$, or the presheaf represented by $[n]$.
Given a simplicial set X, we call X_n its $n-$ simplices, the image $X([n])$, which by Yoneda we conveniently recognize as $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X)$.
An element of X_n is an $n-$simplex of X, which again can be thought of as a morphism $\Delta^n \to X$.
We now introduce coface and codegeneracy maps.
Fixing n and $0 \leq i \leq n$ there is a clear natural transformation

$$\partial_i^n : \Delta^{n-1} \to \Delta^n$$

induced by the unique strictly order preserving map $[n-1] \to [n]$ skipping the index i.
Similarly, there is just one (order preserving, but keep in mind this is always imposed by our definition of the $\mathbf{\Delta}$ category) surjective map $[n+1] \to [n]$ which takes the value i twice: this induces a natural transformation

$$\sigma_i^n : \Delta^{n+1} \to \Delta^n.$$

In fact, it is a very easy exercise to see that all maps $[n] \to [m]$ can be obtained via the composition of the aforementioned coface and codegeneracy maps.
More is true.

Proposition 1.8. *The following relations hold:*

$$\begin{aligned} \partial_j^{n+1}\partial_i^n &= \partial_i^{n+1}\partial_{j-1}^n && i<j \\ \sigma_j^n\sigma_i^{n+1} &= \sigma_i^n\sigma_{j+1}^{n+1} && i\leq j \\ \sigma_j^{n-1}\partial_i^n &= \begin{cases} \partial_i^{n-1}\sigma_{j-1}^{n-2} & i<j \\ 1_{\Delta^{n-1}} & i\in\{j,j+1\} \\ \partial_{i-1}^{n-1}\sigma_j^{n-2} & i>j+1 \end{cases} \end{aligned}$$

and they completely determine $\mathbf{\Delta}$ as a category.

Just for the sake of clarity, let me explain what we mean for a category to "be completely determined by the relations above".
If we take the free category over the graph with vertex $[n]$'s and edges ∂_i^n's and σ_i^n and quotient the relations above, we obtain a category which is isomorphic to $\mathbf{\Delta}$.
I want to strongly point out what this is really telling us about simplicial sets. It is telling us that a simplicial set X essentially consists of just objects X_n, face maps $d_n^i : X_n \to X_{n-1}$ and degeneracy maps $s_n^i : X_n \to X_{n+1}$ which satisfy similar relations as above (notice

[3] To be read as "at the beginning of my understanding of certain mathematical concepts such as this".

the contravariance here).
Topologically speaking, it is fairly clear what is happening: we are just calling "$n-$simplices of X" the part of this abstract entity X which is constructed using $n-$dimensional triangles and we just have to specify how higher and lower dimensional triangles get identified "on the boundaries".
Let me just make a quick remark: thinking of CW complexes usually the notation X_n means "what is constructed using triangles of dimensions **up** n" (rather than just dimension n). I think you can convince yourself that we are never imposing non-degeneracy, so this notation (and the intuition behind) certainly holds. We will, however, give a nice categorical definition of what we *really* want to call **skeleton**.
The following is an easy proposition.

Proposition 1.9. *Any map $\Delta^n \xrightarrow{f} \Delta^m$ factors as $f = i \circ \pi$ where $\pi : \Delta^n \to \Delta^p$ is an epimorphism that admits a section and $i : \Delta^p \to \Delta^n$ is a monomorphism.*

Proof. Essentially just factor through the image of the morphism f as a set-morphism and just take care of the order of the vertex. ☐

Let me now expose, again following Lurie [5], a prototypical example of simplicial sets. We can consider the following object:

$$|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} |\ x_i \geq 0, \sum x_i = 1\}.$$

This is what is typically known as geometric $n-$simplex or standard geometric simplex. Clearly a map $[n] \to [m]$ induces an affine map $|\Delta^n| \to |\Delta^m|$ by simply choosing where to send each coordinate (and summing them if the images overlap).
More clearly:

$$|f|(x_0, \dots, x_n) = \left(\sum_{j \in f^{-1}(i)} x_j \right)_{i=0,m}.$$

This construction yields a functor $\mathbf{sSet} \to \mathbf{Top}$. This functor induces a functor $\mathbf{Top} \to \hat{\boldsymbol{\Delta}} = \mathbf{sSet}$ given by evaluation

$$\mathbf{Top} \ni X \mapsto Sing(X) = ([n] \mapsto \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X)) \in \mathbf{sSet}$$

called **singular chain**. This is a _very fundamental construction in homotopy theory and it is the first brick of a skyscraper called **Homology Theory**. There are lots and lots of references such as [6], [3].
However, let us proceed. We can apply Kan's Theorem (1.6): this evaluation functor has a left adjoint, the *geometric realization* functor $\mathbf{sSet} \to \mathbf{Top}$.
This is (apparently) a very non-explicit way of describing this geometric realization but I think it is useful thinking of it like that: it automatically confirms our intuition: from the point of view of morphism (which in some sense - again by Yoneda - suffices for most applications) we can really think about 0-simplexes as points of a topological space, 1-simplexes as paths and so on.
For a more "hand-on" approach you can read my bachelor thesis where I effectively use all of these functors.
Now we state a new definition, just to complicate things. It won't be useful for the rest of the article but at least it gives a more general way to describe the "skeletons" of certain objects in certain categories.

Definition 1.10. An *Eilenberg-Zilber* category is the datum of (A, A_+, A_-, d) of a small category A and subcategories $A_, A_+$ together with a function $Ob(A) \to \mathbb{N}$ such that:

- any iso of A is contained in both A_-, A_+ and the function d does not distinguish isomorphic objects;
- if $a \to a'$ is a non-identity morphism in A_+ then $d(a) < d(a')$, and the inequality is reversed if the non-identity morphism is in A_-;
- any morphism $u : a \to b$ in A has a unique factorization as $a \to c \to b$ where the first is in A_- and the second one is in A_+;
- every morphism in A_- has a section in A, moreover if two morphism in A_- have the same set of sections then they are equal.

Notice the similarity with the model structures.
Given an **EZ**−category A, an object $a \in A$ with $d(a) = n$ is called $n-$dimensional object. Clearly, from all we've learnt, $\mathbf{\Delta}$ is an EZ-cat taking as degree $[n] \mapsto n$ and as subcategories the ones spanned by the epimorphisms and monomorphisms.

Definition 1.11. Given X a presheaf over an EZ-cat A, a section $x \in Xa$ is called *degenerate* if there exist a morphism $a \xrightarrow{\sigma} a'$ in A with $d(a') < d(a)$ and a section $y \in Xa'$ such that $\sigma^*(y) = x$.
We denote by $Sk_n(X)$ (or its *n-skeleton*) the maximal subpresheaf of X such that for any $m > n$, any sections of $Sk_n(X)$ over an m-dimensional object is degenerate.

Let me now state, without proving, some general results about EZ-cats. Again, the setting is X presheaf over an EZ-cat A.

Lemma 1.12 (Eilenberg-Zilber). *Let $x \in X_a$ be a section of X. There exist a unique decomposition (σ, y) of x such that σ is in A_- and y is non-degenerate.*

Theorem 1.13. *Let $X \subset Y$ be presheaves over A. For any non negative n there is a canonical pushout square*

$$\begin{array}{ccc} \coprod_{y \in \Sigma} \partial h_a & \longrightarrow & X \cup Sk_{n-1}(Y) \\ \big\downarrow & & \big\downarrow \\ \coprod_{y \in \Sigma} \partial h_a & \longrightarrow & X \cup Sk_n(Y) \end{array}$$

where Σ denotes the set of non-degenerate sections of Y of the form $y : h_a \to Y$ which do not belong to X and such that $d(a) = n$. Here $\partial h_a = Sk_{d(a)-1}(h_a)$.

1.3 ... and nerves

Consider the full and faithful inclusion functor

$$i : \mathbf{\Delta} \to Cat$$

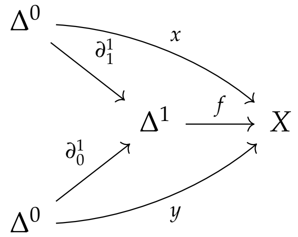
and consider the induced evaluation

$$N : Cat \to \mathbf{sSet}$$

mapping a category C to its *nerve* $([n] \mapsto \mathrm{Hom}_{Cat}([n], C))$.
Again we could construct it a bit more explicitly: the nerve of a category is a simplicial set X having X_n consisting of the n-tuples of composable morphisms. The face and degeneracy consist of inserting an identity morphism (a loop) at the i-th step of the chain and composing the i and $i+1$'th morphism respectively (to be more precise, the first and last morphisms get cancelled).
By what we've already said these informations completely determine the simplicial set X.
By applying Kan's Theorem (1.6) we obtain the existence of a left adjoint functor $i_!$ which will be denoted as $\tau : \mathbf{sSet} \to Cat$.
Again, in my bachelor thesis one can find lots of applications of this functor, however in this article we will use the nerve functor (and its left adjoint) as a way to use simplicial formalism on generic category.
In fact, simplicial sets are very similar to category: we can define "objects" and "morphisms" of a simplicial sets X by simply looking at points and paths on it in the following way.

Definition 1.14. A map of simplicial sets $\Delta^0 \to X$ is an object of X, while a map $\Delta^1 \to X$ is a morphism of X, with source $\Delta^0 \xrightarrow{\partial^1_1} \Delta^1 \to X$ and target $\Delta^0 \xrightarrow{\partial^1_0} \Delta^1 \to X$.

We will shortly indicate with the 'diagram' $x \xrightarrow{f} y$ the datum of



encoding a morphism in a simplicial set X.
Given a finite totally ordered set E, which can clearly be thought of as a category (E again by abusing notation) we put $\Delta^E = N(E)$ and notice[4] that an enumeration of E gives us an isomorphism $\Delta^E \cong \Delta^{|E|}$.

Definition 1.15. The boundary of the standard n-simplex is defined as

$$\partial\Delta^n = \bigcup_{E \subsetneq [n]} \Delta^E.$$

Definition 1.16. Similarly, we call *i-th horn*, for $0 \le i \le n$

$$\Lambda^n_i = \bigcup_{i \in E \subsetneq [n]} \Delta^E.$$

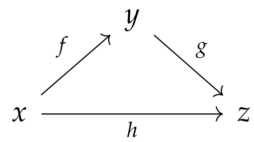
Definition 1.17. We call *spine* of a n-simplex

$$Sp^n = \bigcup_{0 \le i < n} \Delta^{\{i,i+1\}}.$$

[4]In an obvius way, this is a generalization of the previously used notation: $\Delta^{[n]} = \Delta^n$

Now we come to a couple of examples and a fundamental notion of "commutative triangle" in a simplicial set.
This is, in general, kind of a (re)lax commutativity with respect to the ordinary commutativity in the categorical sense, but we will se that the nerve functor is able to control such property.
In retrospect, for the same reason that homotopy theory is a very functional and highly awarding theory in topology, this lax(er) notion will bring beautiful results almost immediately.

For us, a **triangle**[5] in a simplicial set X is the datum of a morphism $\partial\Delta^2 \xrightarrow{\phi} X$. Namely, remembering that $\partial\Delta^2 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,2\}}$ we can identify the map ϕ with the datum of a triple (f, g, h), where $f : \Delta^{\{0,1\}} \to X,\ g : \Delta^{\{1,2\}} \to X,\ h : \Delta^{\{0,2\}} \to X$, with the target of f equal to the source of g and the target of h equal to the target of h, whose source is the sourse of f.
In a shorter way, a triangle in X is, with the notations used above, a diagram of the form



Similarly, the datum of a map $Sp^2 \to X$ is a diagram of the form

$$x \xrightarrow{f} y \xrightarrow{g} z$$

and most importantly, note that $Sp^2 = \Lambda^2_1$.
Now we come to the notion of commutativity.

Definition 1.18. A **commutative triangle** in X is a triangle (f, g, h) such that there exist a simplicial set morphism $\varphi : \Delta^2 \to X$ with the property that $\varphi \restriction \partial\Delta^2 \cong (f, g, h)$.

With the aforementioned definition, we say that given two composable[6] morphisms f, g in X, h is **a** composition if (f, g, h) is a commutative triangle.
Note that composition is not necessarily unique.
Let's continue the exploration on the nerve functor.
Observe that given a **small** category $\mathcal{C}$, then the objects and morphisms of $X = N(\mathcal{C})$ as a simplicial sets coincide in a natural way with the objects and arrows of $\mathcal{C}$in the categorical sense.
More is true: commutative triangles in X correspond precisely to the "real" commutative triangles in $\mathcal{C}$.
What is true is that, in a category if f, g is a composable pair then there exists at least a commutative triangle $f, g, g \circ f$ and this is more important than the other with the same "spine".
This "correspondence" between pair of composable morphisms and triangles, turns out to induce a **bijection**

$$\mathrm{Hom}(\Delta^2, N(\mathcal{C})) \xrightarrow{\cong} \mathrm{Hom}(Sp^2, N(\mathcal{C})).$$

[5]Note, we are not asking commutativity in any way. That will come out later.
[6]So the target of f is the source of g

Proof. This is easily proved: $N(\mathcal{C})_2$ is made exactly by $\bullet \to \bullet \to \bullet$ and by using f, g as first and second morphism we obtain the inverse map $\mathrm{Hom}(Sp^2, N(\mathcal{C})) \to \mathrm{Hom}(\Delta^2, N(\mathcal{C}))$ to the above restriction. □

Theorem 1.19. *Given a small cat $\mathcal{C}$, the restriction map give rise to a bijection*

$$Hom(\Delta^n, N(\mathcal{C})) \overset{\cong}{\to} Hom(Sp^n, N(\mathcal{C}))$$

Proof. I just wanted to insert this proof paragraph, the proof is the literal copy of the above one and I will not copy-paste it. □

This is very important: in the small settings, understanding the "homotopy type" (in some sense) of a nerve is the same as understanding its arrow.
To give a map from $X \to N(\mathcal{C})$ it suffices to give a map $X_1 \to Mor(\mathcal{C})$ such that the identity of all objects of X goes to an identity and that sends commutative triangles (in the simplicial setting) to compositions.

Proposition 1.20. *For any simplicial set X, the inclusion $Sk_2(X) \subset X$ induces an isomorphism of category $\tau(Sk_2(X)) \cong \tau(X)$.*

Definition 1.21 (Grothendieck-Segal condition)**.** A simplicial set X is said to satisfy the Grothendieck-Segal condition if the inclusion of the n-th spine in the standard n-simplex induces a bijection $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(Sp^n, X)$ for all $n \geq 2$.

Thus, we said that the nerve of a category respects the Grothendieck-Segal condition. What we actually proved, said in other terms, is that the functor $N(\bullet) : Cat \to \mathbf{sSet}$ is fully faithful on the subcategory of small categories: it induces bijections

$$\mathrm{Hom}_{Cat}(A, B) \cong \mathrm{Hom}_{\mathbf{sSet}}(N(A), N(B))$$

for all small A, B.
More is true and fascinatingly simple to prove.

Theorem 1.22. *Given a simplicial set X, the following are equivalent:*

- *$X \cong N(\mathcal{C})$ for some small category $\mathcal{C}$;*
- *the unit map $X \to N(\tau(X))$ is invertible;*
- *X satisfies the Grothendieck-Segal condition.*

Now another restatement of the preceeding result. The proof is a bit convoluted, namely there are a couple of cases to be taken care of separately, but I decided the proof is not so important or enlightening. It - however - certainly reflects the adaptability of the theory we are discovering.

Theorem 1.23. *A simplicial set X satisfies the Grothendieck-Segal condition if and only if the inclusion of its k-th horn in the n-th standard simplex induces a bijection*

$$Hom(\Delta^n, X) \to Hom(\Lambda^n_k, X)$$

for all $n \geq 2, 0 < k < n$.

2 Definition of ∞-cat(egorie)s

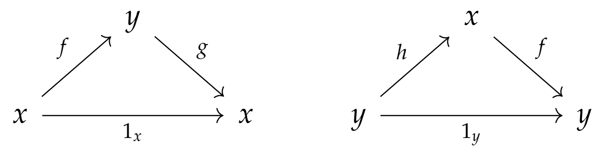
Now the moment we were (hopefully) waiting. Let me restate the definition given in the absolute first page of the article.

Definition 2.1. An ∞-category is a simplicial set X such that the inclusion-induced maps

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$$

are surjective for any $n \geq 2,\ 0 < k < n$.

Definition 2.2. If there exist f, g, h such that



commute, then f is said to be invertible.

Definition 2.3. An ∞-groupoid is an ∞-cat where each morphism is invertible.

Definition 2.4. A Kan complex is an ∞-cat which respect the horn-filling condition for any $n \geq 1,\ 0 \leq k \leq n$.

Hence the name of *Weak Kan complexes* which is sometimes used when referring to ∞-cats.
Note that the horn-filling condition for $n = 2$ implies that every pair of composable morphism of X admit in fact a composition in X: simply consider $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ in X and realize this is the 2-horn Λ^2_1. Filling the horn means there exists an extension starting from Δ^2: then its restriction to $\Delta^{\{0,2\}} \subset \Delta^2$ is a composition of f and g.
This composition in non unique (the horn filling condition states a surjectivity, not an isomorphism as the Grothendieck-Segal condition) but this laxer structures are still of interests because of the "higher coherence" they encode.
The remark between parenthesis actually furnishes the following:

Proposition 2.5. *Given a small category $\mathcal{C}$, its nerve $N(\mathcal{C})$ is an ∞-cat.*

Proof. By Theorem (1.22) we know that the simplicial set $N(\mathcal{C})$ satisfies the Grothendieck-Segal condition hence satisfies the horn-filling condition with uniqueness. □

Proposition 2.6. *A Kan complex is an ∞-groupoid.*

Proof. This is easy. The "enriched" (not in the categorical sense) horn filling conditions tell us that even diagrams such as $\bullet \xleftarrow{f} \bullet \xrightarrow{g} \bullet$ and $\bullet \xrightarrow{f} \bullet \xleftarrow{g} \bullet$ admit extensions to a triangle.
Choosing $g = 1_x$ in the first and $g = 1_y$ in the second one gives us the 2 triangles we are looking for in the definition of invertibility of f.
By the generality of our choices the thesis follows. □

The following is a very deep theorem.

Theorem 2.7 (Equivalence of Kan complexes and ∞-groupoid)**.** *Given X an ∞-groupoid, then X is a Kan complex.*

Let us end this definition section by defining what the opposite of a simplicial set is.

Definition 2.8. Given a non strictly increasing morphism $f : [m] \to [n]$ one can associate to it its **opposite**:

$$\rho(f)(i) = n - f(m-i).$$

What is really happening is that we can think of an increasing function as defined by its jump from left to right: now if we follow the path from right to left we have "negative" jumps: simply negate those and you have a new increasing function.
One can prove that ρ defines a functor $\mathbf{\Delta} \to \mathbf{\Delta}$ and induces

$$\rho^* : \mathbf{sSet} \to \mathbf{sSet}.$$

Given a simplicial set X we define $X^{op} = \rho^*(X)$.

In fact, this behaves well with the only real notion we gave.

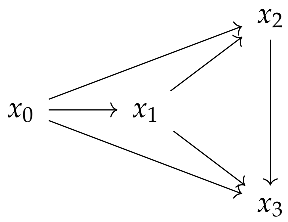
Proposition 2.9. *Given a small category $\mathcal{C}$, the following is a canonical identification:*

$$N(\mathcal{C})^{op} = N(\mathcal{C}^{op}).$$

In the next section we will try to describe how ∞-cat behaves as "real" categories.

3 From an ∞-cat to a cat: a theorem of Boardman & Vogt

For the rest of this section, I will use X to indicate an ∞-cat.
The purpose of this section is givin a somewhat explicit construction of $\tau(X)$: in some sense trying to understand **how** the simplicial structure of X and the higher coherence given by the horn filling conditions, affect its "categorical" structure.
Start by considering the 1-skeleton of Δ^3. Now, with much abstract work, we can return to our root and simply think of the "sticks" you would use to construct a tetrahedron. These are exactly the non degenerate 1-simplices of Δ^3 and they meet at the 0-simplices. So, let x be a map $Sk_1(\Delta^3) \to X$. By what we've said this is simply a diagram of the form



where we are not asking for any type of commutativity[7].
I am sure you can clearly see the 4 triangles in the picture: they are given by applying face maps d^i (namely, forgetting the vertex x_i and the map it touches). The following is a cute lemma which will be fundamental.

Theorem 3.1 (Joyal's Coherence Lemma)**.** *Assume the two 'face' triangles d^0x, d^3x of the map $x : Sk_1(\Delta^3) \to X$ commute.*
Then the other two 'face' triangles either both commute or they both don't commute.

Proof. We just have to prove that, without loss of generality, if d^0x, d^1x, d^3x commute then d^2x commutes as well.
Commutativity of d^0x, d^1x, d^2x implies the existence of suitable extensions $y_0, y_1, y_3 : \Delta^2 \to X$.
Now, d^0x, d^1x, d^3x all have a vertex x_2 in common, so the datum of (y_0, y_1, y_3) guarantees the existence of a map

$$\Lambda^3_2 \to X$$

whose restriction to standard 2-simplices is precisely y_0, y_1, y_3.
Now X is an ∞-cat, so this map admits an extension $\gamma : \Delta^3 \to X$, whose restriction - by construction - on the 1-simplices of $d^2\gamma$ is precisely d^2x. But $d^2\gamma$ is a full face and hence the triangle d^2x is commutative. □

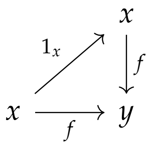
We are almost ready to state the theorem of Boardman and Vogt.
Let us define the following equivalence relations: given a commutative triangle (f, g, h) we will write $gf \sim h$ and state that

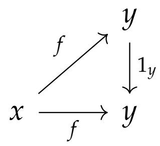
- $f \sim_1 g \Leftarrow f1_x \sim g$;
- $f \sim_2 g \Leftarrow 1_y f \sim g$;
- $f \sim_3 g \Leftarrow g1_x \sim f$;
- $f \sim_1 g \Leftarrow 1_y g \sim f$.

[7] I can't stress this enough and I admit the notation seems confusing at first sight: why would someone draw a diagram if it were not commutative..

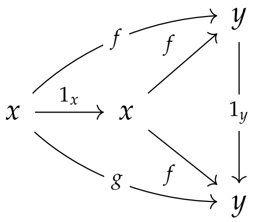
Proposition 3.2. *All the equivalence relations stated above are the same.*

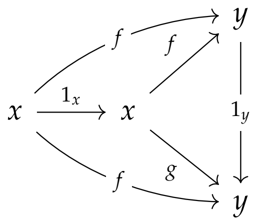
Proof. I will not prove they are in fact equivalence relations. It will be easier to prove afte stating the equality between them.
Instead I will prove they coincide.
The useful remark here is that the diagrams





are always commutative. They come from degenerate 2-simplices obtained via appropriate surjective mappings $\Delta^2 \to \Delta^1$.
Now using Joyal's Coherence Lemma (3.1) to both of the diagrams





and the same in the opposite X^{op}, we obtain exactly what we were looking for.
It is just a matter of explicitly writing everything. □

Definition of homotopy category of a simplicial set

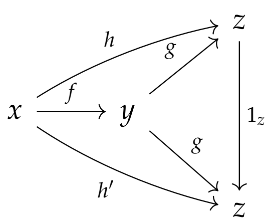
Consider the following construction.
Given two objects x, y in an ∞-cat X, we can quotient out (as sets) the set $\mathrm{Hom}(x, y)$ via the equivalence relations defined above. We will call this set $\mathrm{Hom}_{ho(X)}(x, y)$.
When trying to give a composition law

$$\mathrm{Hom}_{ho(X)}(x, y) \times \mathrm{Hom}_{ho(X)}(y, z) \to \mathrm{Hom}_{ho(X)}(x, z)$$

one should be tempted to just compose some representative

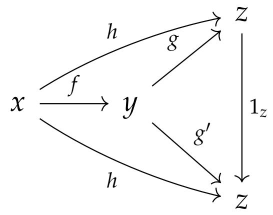
$$[f] \times [g] \to [g] \circ [f] = [h]$$

where h is **a** composition of f, g (which always exists in an ∞-cat as already said).
This works but is not painless.
We should show that none of the above depends on the choice of representatives.
First step: $[h]$ does not depend on h. Let $gf \sim h, gf \sim h'$. Consider the following diagram



Clearly the rightmost triangle commutes (as always) and by hypothesis the top and bottom triangle commutes too. By Joyal's 3.1 all triangles commutes, in particular the

bigger one (which bounds the boundary of the diagram) $\Rightarrow 1_z h \sim h'$ and so $[h'] = [h]$.
Let us show that $[g] = [g']$ and $gf \sim h$ imply $g'f \sim h$ and so the choice of representative of $[g]$ is not important.
Consider the diagram



and again apply Joyal's 3.1 to conclude.
Doing all this in the opposite ∞-cat shows that all of the above does not depend on the choice of representative for $[f]$.
The composition law is then well defined and so produces a new category from X: a category whose objects are precisely the objects of X and whose morphisms are $\mathrm{Hom}_{ho(X)}(\bullet, \bullet)$.
This category is called $ho(X)$ or **homotopy category of** X.
We are now ready to state the main result of this section.

Theorem 3.3 (Boardman & Vogt construction)**.** *There is a unique morphism of simplicial sets*

$$X \to N(ho(X))$$

which is the identity on objects and sends a morphism $[f]$ of X to its class $[f] \in Hom_{ho(X)}(\bullet, \bullet)$. Moreover, this morphism induces an isomorphism of categories

$$\tau(X) \simeq ho(X).$$

Proof. As of now, the proof can be stated very shortly without too many details.
Clearly such a morphism exists and is unique: we've already said that morphisms $X \to N(\mathcal{C})$ are completely determined by $X_1 \to Arr(\mathcal{C})$ which send identities to identities and send commutative triangles to compositions. These conditions are precisely satisfied by our maps, so we have existence and uniqueness of $X \to N(ho(X))$.
More is true: because every $N(\mathcal{C})$ for $\mathcal{C}$ a small category, is actually an ∞-groupoid (hence an ∞-cat), every morphism $X \to N(\mathcal{C})$ has to factor through $X \to N(ho(X)) \to N(\mathcal{C})$ and this factorization is unique (again by looking at maps $X_1 \to Arr(\mathcal{C})$).
Using the fully faithful nature of $N(\bullet)$ we just proved that the objects $\tau(X)$ and $ho(X)$ represent the same functor; hence they are canonically isomorphic. □

Proposition 3.4. *An ∞-cat X is an ∞-groupoid if and only if $\tau(X)$ is a groupoid.*

Proof. Clearly,if X is an ∞-groupoid then every morphism of $ho(X)$ admits an inverse and so is a groupoid. By the isomorphism 3.3 we conclude. □

Now the final homotopy flavor.

Proposition 3.5. *A morphism $f : x \to y$ in an ∞-cat is invertible if and only if there exists a morphism $g : y \to x$ such that gf and fg admit respectively 1_x and 1_y as compositions.*

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