Perverse coherent sheaves and derived equivalences

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Pisa, a.a. 2023/2024

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Contents

Introduction

Noncommutative algebra naturally arises in the study of resolution of singular varieties, as highlighted by the *McKay correspondence*.

This correspondence establishes a bijection between finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and Dynkin quivers. It can be obtained by studying the irreducible representations of such groups.

Definition. Let $G \leq \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup, let ρ be the natural representation of G and $\rho_0, \dots \rho_k$ be the irreducible G-representations. For $i, j \in \{0, \dots, k\}$ let a_{ij} be defined by the decomposition

$$\rho \otimes \rho_i = \bigoplus_j \rho_j^{a_{ij}}.$$

The *McKay quiver* of G has a vertex for each ρ_i and a_{ij} arrows going from vertex i to vertex j.

Theorem (see e.g. [Kir16, Theorem 8.13, Theorem 8.15])**.** The McKay quiver of G is an extended Dynkin quiver of type ADE. Moreover, this assignment establishes a bijection between finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and Dynkin diagrams of type ADE.

As a linear group, G has a natural action on the affine plane $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x,y])$. Let X be the quotient

$$X = \mathbb{C}/G = \mathrm{Spec}(\mathbb{C}[x,y]^G).$$

This is a surface with a unique singular point in the origin. It is known, see e.g. [Kir16, Theorem 12.3], that such a variety admits a minimal resolution of its singularity.

Surprisingly, the McKay quiver of G controls the geometry of the resolution of the singularity of X.

Definition. Let $\pi \colon Y \to X$ be the minimal resolution of the singular point of X. The *resolution graph* of X is a graph with a vertex for every irreducible component of the exceptional divisor $\pi^{-1}(0)$. Two vertices are joined by an edge if the two corresponding components intersect.

Theorem (see e.g. [Kir16, Theorem 12.3])**.** The resolution graph of $X = \mathbb{C}^2/G$ is an Dynkin diagram of type ADE. Moreover, it is exactly the Dynkin diagram whose associated extended Dynkin quiver corresponds to G via the McKay correspondence.

From this example, it seems clear that noncommutative algebra (in this case the representation theory of G) plays a key role in understanding the geometry behind the resolution of singularities, and more in general in birational geometry. In the above case, the crucial objects in understanding the geometry of the resolution were the irreducible representations of G, that is the irreducible objects in $\mathrm{mod}(\mathbb{C}[G])$.

Inspired by this example, a typical approach in birational geometry is to try and find an abelian category (possibly a category of modules over an algebra), which provides insight into the geometry of the morphism under investigation.

The typical approach is to consider a birational morphism

$$f\colon Y \to X$$

between Noetherian schemes over a field k. By assuming suitable hypotheses on f, we may construct a k-algebra A (usually noncommutative) in such a way that its module category $\mathrm{mod}(A)$ captures information about the geometry of f. In even more generality, we may try and construct an abelian category $\mathcal{A}$ encoding geometrical information about the morphism f.

The work presented in this thesis falls under the aforementioned approach. Starting from a morphism $f\colon Y \to X$, we construct two abelian categories, called *perverse coherent sheaves* categories, which capture information about the geometry of the morphism. In the case where the base X is affine, these are equivalent to categories of modules over a k-algebra.

The main reference that guides the content of this thesis is Van den Bergh's paper [Van04]. We now give a brief overview of the situation presented in [Van04], which will be studied in detail in Chapter 2.

We consider a projective birational morphism

$$f\colon Y \to X$$

between Noetherian equidimensional schemes over a field k, assuming the existence of a point $p \in X$ such that the fiber $f^{-1}(p) = C$ is a curve contained in Y, with f being an isomorphism outside C. Additionally, we assume that $\mathbf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$. A prototypical example of this scenario is the resolution of rational singularities of surfaces.

Using direct images of the morphism f, we define two torsion pairs $(\mathcal{T}_p, \mathcal{F}_p)$, for $p = -1, 0$, on the bounded derived category of coherent sheaves $D^b(\mathrm{coh}(Y))$. By applying the process of tilting to these torsion pairs, we obtain two abelian categories

$$\mathrm{Per}^p(Y/X) \subseteq D^b(\mathrm{coh}(Y)),$$

referred to as categories of perverse coherent sheaves. These categories are "tilts" of the standard heart of the derived category and encode important information about the morphism f. For example, if f is a resolution of a rational singularity, the perverse coherent categories provide insights into the type of the singularity.

The main result of this thesis establishes an equivalence between the perverse coherent sheaf categories $\mathrm{Per}^p(Y/X)$ and a category of coherent sheaves. More formally, we prove the following theorem.

Theorem A (Theorem 2.4.2)**.** Assume that $\mathcal{P}$ is a local projective generator in $\mathrm{Per}^p(Y/X)$ and let $\mathcal{A} := f_*\mathcal{E}nd_Y(\mathcal{P})$. Then the functors

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_Y(\mathcal{P}, -)\colon D^b(\mathrm{coh}(Y)) \longrightarrow D^b(\mathrm{coh}(\mathcal{A})),$$
$$f^{-1}(-) \overset{\mathbf{L}}{\otimes}_{f^{-1}(\mathcal{A})} \mathcal{P}\colon D^b(\mathrm{coh}(\mathcal{A})) \to D^b(\mathrm{coh}(Y))$$

define inverse equivalences of triangulated categories. These equivalences map the perverse t-structure of $D^b(\mathrm{coh}(Y))$ to the canonical t-structure of $\mathrm{coh}(\mathcal{A})$. Therefore, they restrict to equivalences between $\mathrm{Per}^p(Y/X)$ and $\mathrm{coh}(\mathcal{A})$.

This result establishes a deep connection between the derived category of coherent sheaves on Y and the category of coherent sheaves on a certain sheaf of algebras $\mathcal{A}$, which is the pushforward of the endomorphism algebra of a projective generator.

A crucial part of the proof is an analogous statement for the case where the base scheme $X = \operatorname{Spec} R$ is affine. In this situation, the categories of perverse coherent sheaves are equivalent to categories of modules over a k-algebra. This more algebraic setting allows us to leverage tools from representation theory and homological algebra, which simplify the analysis. The corresponding affine version of the main theorem is as follows.

Theorem B (Theorem 2.3.20)**.** Suppose that the base scheme $X = \operatorname{Spec} R$ is affine. Let $\mathcal{P}$ be a projective generator in $\operatorname{Per}^p(Y/X)$ and let $A := \operatorname{End}_Y(\mathcal{P})$. To distinguish the R-module structure and the A-module structure on $\mathcal{P}$, we denote the latter by ${}_A\mathcal{P}$. Then the functors

$$\mathbf{R}\operatorname{Hom}_Y({}_A\mathcal{P}, -)\colon D^b(\operatorname{coh}(Y)) \longrightarrow D^b(A),$$
$$- \overset{\mathbf{L}}{\otimes}_A {}_A\mathcal{P}\colon D^b(A) \to D^b(\operatorname{coh}(Y))$$

define inverse equivalences of triangulated categories. These equivalences map the perverse t-structure of $D^b(\operatorname{coh}(Y))$ to the canonical t-structure of $D^b(A)$. Therefore, they restrict to equivalences between $\operatorname{Per}^p(Y/X)$ and $\operatorname{mod}(A)$.

The significance of these theorems relies on the existence of a projective generator, as established by the following results.

Proposition C (Proposition 2.4.3)**.** Assume that X is quasi-projective over a Noetherian ring S. Then there exists a local projective generator $\mathcal{P}$ for $\operatorname{Per}^{-1}(Y/X)$, such that the dual $\mathcal{P}^\vee$ is a local projective generator for $\operatorname{Per}^0(Y/X)$.

Proposition D (Proposition 2.3.15)**.** Suppose $X = \operatorname{Spec} R$ is affine. Then, there exists a vector bundle $\mathcal{P}$ which is a projective generator in $\operatorname{Per}^{-1}(Y/X)$ and whose dual $\mathcal{P}^\vee$ is a projective generator in $\operatorname{Per}^0(Y/X)$.

Moreover, we are able to give an explicit characterization of the projective generators in $\operatorname{Per}^p(Y/X)$, thus allowing a deeper study of the algebra A. This is highlighted in the last part of the thesis, where we we perform some explicit computations.

By supposing that $X = \operatorname{Spec} R$ is the spectrum of a Noetherian complete local k-algebra with residue field k (the *formal case*), we are able to make a canonical choice of a projective generator and find properties of its endomorphisms algebra $\operatorname{End}_Y(\mathcal{P})$.

Notice that Proposition C and Proposition D establish a form of duality between the two perverse categories. In the affine case, we obtain

$$\operatorname{Per}^{-1}(Y/X) \simeq \operatorname{mod}(\operatorname{End}_Y(\mathcal{P})) \quad \text{and} \quad \operatorname{Per}^0(Y/X) \simeq \operatorname{mod}(\operatorname{End}_Y(\mathcal{P}^\vee)).$$

Thanks to Theorem B, it becomes clear that the endomorphism rings of projective objects in linear categories are of great importance to the study of birational geometry. The module categories of such rings can be studied through the tools of Morita theory. In Appendix A we review the basics of Morita theory for finite dimensional k-algebras.

Outline of the thesis

The thesis is organized as follows.

Chapter 1: Torsion pairs, t-structures and tilting

This chapter provides the foundational background on torsion pairs and t-structures in abelian and derived categories. Here, we set up the necessary tools and concepts that are used throughout the thesis to define perverse coherent sheaves.

We introduce tilting theory and discuss its significance in constructing new abelian categories from derived categories. This discussion is primarily based on the work of Happel, Reiten, and Smalø, [HRS96], which builds upon the foundational theory introduced by Beilinson, Bernstein, and Deligne, [BBD82].

Chapter 2: Perverse coherent sheaves

This chapter constitutes the core of the thesis, where we closely follow [Van04]. We first introduce the geometric setting and define the torsion pairs $(\mathcal{T}_p, \mathcal{F}_p)$. Once the perverse categories $\mathrm{Per}^p(Y/X)$ are defined and their basic properties studied, we move on to proving Theorem B and Theorem A.

In the last part of the chapter we make some explicit computations. First we study the *formal case*, where we are able to express more explicitly the projective generators. Finally, we apply the proven equivalence to the study of certain birational morphisms.

Appendix A: Morita Theory

Two k-algebras A and B are said to be Morita equivalent if their module categories $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ are equivalent. Therefore, to study a k-algebra A in the context of Morita theory is to study the properties of its module category. In the appendix, we review the basic results concerning Morita theory of finite-dimensional k-algebras, following mainly [AC20]. The combinatorial aspects of Morita theory are particularly evident through the study of quivers associated with these algebras.

Chapter 1

Torsion pairs, t-structures, and tilting

In this chapter, we introduce the notions of torsion pairs of an abelian category $\mathcal{A}$ and t-structures of the derived category $D(\mathcal{A})$. Furthermore, we show the relation between torsion pairs and t-structures. We follow [HRS96].

1.1 Torsion pairs, t-structures, and tilting

Let $\mathcal{A}$ be an abelian category. We denote by $C(\mathcal{A})$ the category of complexes in $\mathcal{A}$ and by $C^{+,b}(\mathcal{A})$, $C^{-,b}(\mathcal{A})$ the categories of complexes in $\mathcal{A}$, which are respectively bounded below/above and have bounded cohomology. Similarly, denote by $K(\mathcal{A})$, $K^{+,b}(\mathcal{A})$, $K^{-,b}(\mathcal{A})$ the corresponding homotopy categories.

Let $D(\mathcal{A})$ be the derived category of $\mathcal{A}$ and $D^{+}(\mathcal{A})$, $D^{-}(\mathcal{A})$, $D^{b}(\mathcal{A})$ be the subcategories of complexes with cohomology bounded respectively below, above, or both.

Remark 1.1.1. The derived category $D^b(\mathcal{A})$ endowed with the natural translation functor is a triangulated category.

Definition 1.1.2. A *torsion pair* is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$, which satisfy the following properties.

(a) $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(b) Every object $X \in \mathcal{A}$ fits into a *torsion exact sequence*

$$0 \to t(X) \to X \to X/t(X) \to 0, \tag{1.1}$$

with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.

The category $\mathcal{T}$ is called *torsion class* and its objects are called *torsion objects*. The category $\mathcal{F}$ is called *torsion free class* and its objects are called *torsion free objects*.

Proposition 1.1.3. The following properties are satisfied

(a) If $X \in \mathcal{A}$ is an object such that $\mathrm{Hom}_{\mathcal{A}}(T, X) = 0$ for all $T \in \mathcal{T}$ then X belongs in $\mathcal{F}$.

(b) If $X \in \mathcal{A}$ is an object such that $\mathrm{Hom}_{\mathcal{A}}(X, F) = 0$ for all $F \in \mathcal{F}$ then X belongs in $\mathcal{T}$.

(c) The subcategories $\mathcal{T}, \mathcal{F}$ are closed under extensions. Moreover, $\mathcal{T}$ is closed under quotient objects, while $\mathcal{F}$ is closed under subobjects.

Proof. Properties (a) and (b) follow trivially from the torsion exact sequence (1.1).

We prove (c) for the subcategory $\mathcal{T}$, the proof for $\mathcal{F}$ is analogous. Let

$$0 \to X' \to X \to X'' \to 0$$

be a short exact sequence in $\mathcal{A}$. For each object $F \in \mathcal{F}$, we may act on the sequence via the functor $\mathrm{Hom}_{\mathcal{A}}(-, F)$ and get the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(X'', F) \to \mathrm{Hom}_{\mathcal{A}}(X, F) \to \mathrm{Hom}_{\mathcal{A}}(X', F).$$

If X belongs to $\mathcal{T}$, then $\mathrm{Hom}_{\mathcal{A}}(X, F) = 0$ for all $F \in \mathcal{F}$ and therefore $\mathrm{Hom}_{\mathcal{A}}(X'', F) = 0$ for all $F \in \mathcal{F}$. Using (b) we get that $\mathcal{T}$ is closed under quotient objects.

Similarly, if X' and X'' belong to $\mathcal{T}$, then $\mathrm{Hom}_{\mathcal{A}}(X', F) = \mathrm{Hom}_{\mathcal{A}}(X', F) = 0$ for all $F \in \mathcal{F}$. Using (b) we get that $\mathcal{T}$ is closed under extensions. $\square$

Let $\mathcal{C}$ be a triangulated category. For $X \in \mathcal{C}$ denote the translate of X by $T(X) = X[1]$ and recursively $T^n(X) = X[n]$.

Definition 1.1.4. A *t-structure* on $\mathcal{C}$ is a pair $\left(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}\right)$ of full subcategories of $\mathcal{C}$ such that, setting $\mathcal{C}^{\geq n} := \mathcal{C}^{\geq 0}[-n]$ and $\mathcal{C}^{\leq n} := \mathcal{C}^{\leq 0}[-n]$ for $n \in \mathbb{N}$, the following properties are satisfied.

(a) $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for all $X \in \mathcal{C}^{\leq 0},\ Y \in \mathcal{C}^{\geq 1}$.

(b) $\mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0}$.

(c) For all $X \in \mathcal{C}$ there exists a triangle

$$X' \to X \to X'' \to X[1]$$

with $X' \in \mathcal{C}^{\leq 0}$, and $X'' \in \mathcal{C}^{\geq 1}$.

Definition 1.1.5. Let $\left(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}\right)$ be a t-structure on the category $\mathcal{C}$. The *heart* of the t-structure is defined as the full subcategory $\mathcal{H} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$.

Theorem 1.1.6 ([BBD82])**.** The heart $\mathcal{H}$ is an abelian category.

Example 1.1.7. If $\mathcal{C} = D^b(\mathcal{A})$ is the bounded derived category of $\mathcal{A}$, we can define a trivial t-structure on $\mathcal{C}$. Let

$$\mathcal{C}^{\leq 0} := \left\{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \text{ for all } i > 0 \right\},$$
$$\mathcal{C}^{\geq 0} := \left\{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \text{ for all } i < 0 \right\}.$$

On $C(\mathcal{A})$ there is a truncation functor $\tau_{\leq 0}$ defined as follows. If

$$X^\bullet = \cdots X^i \to X^{i+1} \to \cdots \to X^{-1} \to X^0 \to \cdots$$

is an object in $C(\mathcal{A})$, its truncation is

$$\tau_{\leq 0} X^\bullet = \cdots X^i \to X^{i+1} \to \cdots \to X^{-1} \to \ker d_X^0 \to 0 \to 0 \cdots .$$

Denote by $\tau_{\geq 1} X^\bullet$ the quotient $X^\bullet / \tau_{\leq 0} X^\bullet$. Then in $C(\mathcal{A})$ there is a short exact sequence

$$0 \to \tau_{\leq 0} X^\bullet \to X^\bullet \to \tau_{\geq 1} X^\bullet \to 0,$$

which yields a triangle in $D^b(\mathcal{A})$

$$\tau_{\leq 0}X^\bullet \xrightarrow{\mu} X^\bullet \xrightarrow{\pi} \tau_{\geq 1}X^\bullet \to \tau_{\leq 0}X^\bullet[1] \tag{1.2}$$

Since $\tau_{\leq 0}X^\bullet$ belongs to $\mathcal{C}^{\leq 0}$ and $\tau_{\geq 1}X^\bullet$ belongs to $\mathcal{C}^{\geq 0}$, (1.2) proves that $\left(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}\right)$ is a t-structure on $D^b(\mathcal{A})$, called the *trivial t-structure*. Notice that the heart $\mathcal{H}$ of $\left(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}\right)$ is formed by the complexes in $D^b(\mathcal{A})$ which have vanishing cohomology in all non-zero degrees. It coincides with the essential image of the inclusion

$$\mathcal{A} \hookrightarrow D^b(\mathcal{A}).$$

Therefore, the heart of the trivial t-structure on $D^b(\mathcal{A})$ is equivalent to $\mathcal{A}$.

Example 1.1.8. For later reference, we study in more details the properties of the truncation functor $\tau_{\leq 0}$ and introduce a second truncation functor σ.

First notice that the inclusion

$$\mu \colon \tau_{\leq 0}X^\bullet \to X^\bullet$$

yields isomorphisms in cohomology in all degrees $i \leq 0$, while $\tau_{\leq 0}X^\bullet$ clearly has vanishing cohomology in degrees $i > 0$. Therefore, $H^i(\pi)$ is an isomorphism for all $i > 0$ and $\tau_{\geq 1}X^\bullet$ has vanishing cohomology in degrees $i \leq 0$.

Define $\sigma(\tau_{\leq 0}X^\bullet)$ to be the subcomplex

$$\sigma(\tau_{\leq 0}X^\bullet) = \cdots X^i \to X^{i+1} \to \cdots \to X^{-1} \to \operatorname{Im} d_X^{-1} \to 0 \to 0 \cdots,$$

which yields a triangle in $D^b(\mathcal{A})$

$$\sigma(\tau_{\leq 0}X^\bullet) \xrightarrow{i} \tau_{\leq 0}X^\bullet \xrightarrow{\rho} H^0(X^\bullet) \to \sigma(\tau_{\leq 0}X^\bullet)[1].$$

Then $\sigma(\tau_{\leq 0}X^\bullet)$ has vanishing cohomology in degrees $i \geq 0$, and cohomology isomorphic to $H^i(X^\bullet)$ in degrees $i < 0$.

Proposition 1.1.9. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on an abelian category $\mathcal{A}$. Define

$$\mathcal{D}^{\leq 0} := \left\{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i > 0 \text{ and } H^0(X) \in \mathcal{T} \right\},$$
$$\mathcal{D}^{\geq 0} := \left\{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i < -1 \text{ and } H^{-1}(X) \in \mathcal{F} \right\}.$$

Then, the pair $\left(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}\right)$ is a t-structure on $D^b(\mathcal{A})$.

Proof. We need to check that the conditions (a), (b) and (c) in Definition 1.1.4 are satisfied.

Let

$$X \in \mathcal{D}^{\leq 0}$$

and

$$Y \in \mathcal{D}^{\geq 1} = \left\{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X) = 0 \text{ for all } i < 0 \text{ and } H^0(X) \in \mathcal{F} \right\}$$

Suppose that there exists a nonzero morphism $\operatorname{Hom}_{D^b(\mathcal{A})}(X^\bullet, Y^\bullet)$. It is represented by a pair of morphisms (f, s) in $K^b(\mathcal{A})$

$$\begin{tikzcd} & Z^\bullet & \\ X^\bullet \arrow[ur, "f"] & & Y^\bullet, \arrow[ul, "s"'] \end{tikzcd}$$

where s is a quasi isomorphism. Since s is a quasi isomorphism, $Z^\bullet$ belongs to $\mathcal{D}^{\geq 1}$ and f is a nonzero morphism in $\mathrm{Hom}_{K^b(\mathcal{A})}(X^\bullet, Z^\bullet)$. Using the truncation functor $\tau_{\leq 0}$, we find the following commutative diagram in $D^b(\mathcal{A})$, whose lines are triangles in $D^b(\mathcal{A})$

$$\begin{tikzcd}
\tau_{\leq 0}X^\bullet \arrow[r, "\mu"] \arrow[d, "\tau_{\leq 0}f"] & X^\bullet \arrow[r, "\pi"] \arrow[d, "f"] & \tau_{\geq 1}X^\bullet \arrow[r] \arrow[d, "h"] & \tau_{\leq 0}X^\bullet[1] \arrow[d] \\
\tau_{\leq 0}Z^\bullet \arrow[r] & Z^\bullet \arrow[r] & \tau_{\geq 1}Z^\bullet \arrow[r] & \tau_{\leq 0}X^\bullet[1],
\end{tikzcd}$$

where the morphism h exists by the axioms of *triangulated category*. As observed in Example 1.1.8, the complex $\tau_{\geq 1}X^\bullet$ has vanishing cohomology in degrees $i \leq 0$ and cohomology isomorphic to that of the complex $X^\bullet$ in degrees $i > 0$. Since $X^\bullet$ belongs to $\mathcal{D}^{\leq 0}$, this proves that the complex $\tau_{\geq 1}X^\bullet$ is acyclic, hence it is zero in the derived category $D^b(\mathcal{A})$. This proves that μ is is an isomorphism in $D^b(\mathcal{A})$ and therefore $\tau_{\leq 0}f$ is a nonzero morphism in $K^b(\mathcal{A})$.

We now act a further truncation via the functor σ and get the following commutative diagram

$$\begin{tikzcd}
\sigma(\tau_{\leq 0}X^\bullet) \arrow[r] \arrow[d, "\sigma\tau_{\leq 0}f"] & \tau_{\leq 0}X^\bullet \arrow[r] \arrow[d, "\tau_{\leq 0}f"] & H^0(X^\bullet) \arrow[r] \arrow[d, "h'"] & \sigma(\tau_{\leq 0}X^\bullet)[1] \arrow[d] \\
\sigma(\tau_{\leq 0}Z^\bullet) \arrow[r] & \tau_{\leq 0}Z^\bullet \arrow[r, "\rho"] & H^0(Z^\bullet) \arrow[r] & \sigma(\tau_{\leq 0}Z^\bullet)[1].
\end{tikzcd}$$

Again, it follows by the isomorphisms observed in Example 1.1.8 that $\sigma(\tau_{\leq 0}X^\bullet)$ is acyclic, hence it is zero in $D^b(\mathcal{A})$ and therefore ρ is an isomorphism. By hypothesis $H^0(X^\bullet) \in \mathcal{T}$ and $H^0(Z^\bullet) \in \mathcal{F}$, so h' must be the zero morphism. Then $\tau_{\leq 0}f$ must be zero, getting a contradiction. So the axiom (a) of a t-structure is satisfied.

The property (b) is trivial.

We are left to prove property (c). Let $X^\bullet \in D^b(\mathcal{A})$. Since $(\mathcal{T}, \mathcal{F})$ is a torsion pair, in $\mathcal{A}$ there is a short exact sequence

$$0 \to t(H^0(X^\bullet)) \xrightarrow{\mu} H^0(X^\bullet) \xrightarrow{\pi} H^0(X^\bullet)/t(H^0(X^\bullet)) \to 0,$$

with $t(H^0(X^\bullet)) \in \mathcal{T}$ and $H^0(X^\bullet)/t(H^0(X^\bullet)) \in \mathcal{F}$. Consider the following diagram of short exact sequences in $\mathcal{A}$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & t(H^0(X^\bullet)) \arrow[d, "\mu"] & \\
0 \arrow[r] & \operatorname{Im} d_X^{-1} \arrow[r] & \ker d_X^0 \arrow[r] & H^0(X^\bullet) \arrow[r] \arrow[d] & 0 \\
 & & & H^0(X^\bullet)/t(H^0(X^\bullet)) \arrow[d] & \\
 & & & 0. &
\end{tikzcd}$$

By pulling back μ along the horizontal short exact sequence, we get the following commutative diagram of short exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Im} d_X^{-1} \arrow[r, "\mu''"] \arrow[d, equal] & E \arrow[r] \arrow[d, "\mu'"] & t(H^0(X^\bullet)) \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \operatorname{Im} d_X^{-1} \arrow[r] & \ker d_X^0 \arrow[r] \arrow[d] & H^0(X^\bullet) \arrow[r] \arrow[d] & 0 \\
 & & H^0(X^\bullet)/t(H^0(X^\bullet)) \arrow[r, equal] \arrow[d] & H^0(X^\bullet)/t(H^0(X^\bullet)) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

We now use the morphism μ'' to construct $X'^\bullet \in \mathcal{D}^{\leq 0}$. Decompose d_X^{-1} as follows

$$\begin{tikzcd}
X^{-1} \arrow[r, "\rho"] \arrow[rr, bend right, "d_X^{-1}"'] & \operatorname{Im} d_X^{-1} \arrow[r, "i"] & X^0.
\end{tikzcd}$$

Let $\widetilde{d_X^{-1}} := \mu''\rho$ and $X'^\bullet$ be the subcomplex of $X^\bullet$ defined by

$$X'^i = X^i \text{ for } i \leq 1, \quad X'^0 = E, \quad X'^i = 0 \text{ for } i > 0$$
$$d_{X'}^i = d_X \text{ for } i < -1, \quad d_{X'}^{-1} = \widetilde{d_X^{-1}}, \quad d_{X'}^i = 0 \text{ for } i \geq 0.$$

Notice that $H^i(X'^\bullet) = H^i(X^\bullet)$ for $i \leq 1$. Indeed, the case $i < -1$ is trivial, and for the case $i = -1$ it suffices to use the definition of $\widetilde{d_X^{-1}}$. Moreover,

$$H^0(X'^\bullet) = E/\operatorname{Im} \widetilde{d_X^{-1}} = E/\operatorname{Im} \mu'' = t(H^0(X^\bullet)) \in \mathcal{T}$$

and $H^i(X'^\bullet) = 0$ for $i > 0$. This proves that $X'^\bullet$ belongs to $\mathcal{D}^{\leq 0}$.

Set $X''^\bullet := X^\bullet/X'^\bullet$. Then, the following is a triangle in $D^b(\mathcal{A})$

$$X'^\bullet \to X^\bullet \to X''^\bullet \to X'^\bullet[1]. \tag{1.3}$$

In order to conclude the proof, we need to show that $X''^\bullet$ belongs to $\mathcal{D}^{\geq 1}$. It follows from the definition of $X''^\bullet$ that $H^i(X''^\bullet) = 0$ for $i < 0$. Moreover, the long exact cohomology sequence obtained from (1.3) yields

$$0 = H^{-1}(X''^\bullet) \to H^0(X'^\bullet) = t(H^0(X^\bullet)) \to H^0(X^\bullet) \to H^1(X''^\bullet) \to H^1(X'^\bullet) = 0.$$

This proves that $H^1(X''^\bullet) = H^0(X^\bullet)/t(H^0(X^\bullet))$ belongs to $\mathcal{F}$, thus concluding the proof. $\square$

Corollary 1.1.10. Let $\mathcal{A}$ be an abelian category and $(\mathcal{T}, \mathcal{F})$ a torsion pair on $\mathcal{A}$. Then

(a) The full subcategory

$$\mathcal{B} := \left\{ X^\bullet \in D^b(\mathcal{A}) \mid H^i(X^\bullet) = 0 \text{ for all } i \neq 0, -1, \ H^0(X^\bullet) \in \mathcal{T}, \ H^{-1}(X^\bullet) \in \mathcal{F} \right\}$$

is an abelian category.

(b) Let $\mathcal{X} := \mathcal{F}[1]$ and $\mathcal{Y} := \mathcal{T}$ be full subcategories of $\mathcal{B}$. Then $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{B}$.

(c) For all $X, Y \in \mathcal{B}$ there are natural isomorphisms

$$\mathrm{Hom}_{D^b(\mathcal{B})}(X, Y[n]) \simeq \mathrm{Hom}_{D^b(\mathcal{A})}(X, Y[n])$$

for $n = 0, 1$.

Proof. The statement (a) is just a consequence of Proposition 1.1.9 and Theorem 1.1.6.

We now prove (b). In particular, we need to check that the two axioms defining a torsion pair a satisfied. Let $X \in \mathcal{X} = \mathcal{F}[1]$ and $Y \in \mathcal{Y} = \mathcal{T}$. So there exists $F \in \mathcal{F}$ such that $X = F[1]$. Then

$$\mathrm{Hom}_{\mathcal{B}}(X, Y) = \mathrm{Hom}_{D^b(\mathcal{A})}(F[1], Y) = \mathrm{Hom}_{D^b(\mathcal{A})}(F, Y[-1]) = 0,$$

so the first condition is satisfied.

For the second part, notice first that every object in $\mathcal{B}$ is isomorphic to a two-term complex. Indeed, let $Z^\bullet \in \mathcal{B}$. By definition of $\mathcal{B}$ we get that $H^i(Z^\bullet) = 0$ for all $i > 0$. Therefore, $Z^\bullet$ is isomorphic to $\tau_{\leq 0} Z^\bullet$ in $D^b(\mathcal{A})$. Let $U^\bullet$ be the subcomplex of $\tau_{\leq 0} Z^\bullet$ defined by

$$U^i = Z^i \text{ for } i < -1, \quad U^{-1} = \mathrm{Im}\, d_Z^{-2}, \quad U^i = 0 \text{ for } i \geq 0.$$

Then $U^\bullet$ is an acyclic complex, hence it is zero in $D^b(\mathcal{A})$. Therefore,

$$Z^\bullet \simeq \tau_{\leq 0} Z^\bullet \simeq \tau_{\leq 0} Z^\bullet / U^\bullet =: \widetilde{Z^\bullet}$$

and $\widetilde{Z^\bullet}$ is a complex that is nonzero only in degrees -1 and 0.

In conclusion, in order to prove condition (b), we may assume without loss of generality that $Z^\bullet$ is concentrated in degrees -1 and 0, i.e.

$$Z^\bullet = \ldots 0 \to Z^{-1} \xrightarrow{d_Z^{-1}} Z^0 \to 0 \ldots,$$

with $H^{-1}(Z^\bullet) = \ker d_Z^{-1} \in \mathcal{F}$ and $H^0(Z^\bullet) = \mathrm{coker}\, d_Z^{-1} \in \mathcal{T}$. Then, the following diagram represents the torsion exact sequence of $Z^\bullet$ in $\mathcal{B}$

$$\begin{tikzcd}
\cdots & 0 \arrow[r] & H^{-1}(Z^\bullet) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 & \cdots \\
\cdots & 0 \arrow[r] & Z^{-1} \arrow[r, "d_Z^{-1}"] \arrow[d] & Z^0 \arrow[r] \arrow[d] & 0 & \cdots \\
\cdots & 0 \arrow[r] & 0 \arrow[r] & H^0(Z^\bullet) \arrow[r] & 0 & \cdots
\end{tikzcd}$$

This concludes the proof of (b).

The statement (c) is [BBD82, Remark 3.1.17]. $\square$

1.2 Properties of torsion pairs

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Following the notation of Corollary 1.1.10, we denote the pair $(\mathcal{B}; (\mathcal{X}, \mathcal{Y}))$ as $\Phi(\mathcal{A}; (\mathcal{T}, \mathcal{F}))$. We say that $\Phi(\mathcal{A}; (\mathcal{T}, \mathcal{F}))$ is obtained from $\mathcal{A}$ by *tilting* with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$.

Definition 1.2.1.

(a) The subcategory $\mathcal{T}$ is a *tilting torsion class* if $\mathcal{T}$ is a cogenerator in $\mathcal{A}$. Namely, for all $X \in \mathcal{A}$ there exists an object $T_X \in \mathcal{T}$ with a monomorphism $\mu_X \colon X \to T_X$.

(b) The subcategory $\mathcal{F}$ is a *cotilting torsion free class* if $\mathcal{F}$ is a generator in $\mathcal{A}$. Namely, for all $X \in \mathcal{A}$ there exists an object $F_X \in \mathcal{F}$ with a epimorphism $\pi_X \colon F_X \to X$.

Proposition 1.2.2. The following properties hold.

(a) The torsion class $\mathcal{T}$ is tilting in $\mathcal{A}$ if and only if the torsion free class $\mathcal{Y}$ is cotilting in $\mathcal{B}$.

(b) The torsion free class $\mathcal{F}$ is cotilting in $\mathcal{A}$ if and only if the torsion class $\mathcal{X}$ is tilting in $\mathcal{B}$.

Proof. We prove (a), the proof of (b) is dual.

Suppose first that $\mathcal{T}$ is a tilting torsion class in $\mathcal{A}$ and fix $X^\bullet$ in $\mathcal{B}$. We need to find an object $E \in \mathcal{Y} = \mathcal{T}$ with an epimorphism $E \to X^\bullet$ in $\mathcal{B}$. As show in the proof Corollary 1.1.10, we may assume without loss of generality that $X^i = 0$ for $i \neq -1, 0$. By the assumption that $\mathcal{T}$ is tilting, in $\mathcal{A}$ we find a monomorphism $X^{-1} \to T_0$, with $T_0 \in \mathcal{T}$, which may be completed to a short exact sequence

$$0 \to X^{-1} \to T_0 \to T_1 \to 0. \tag{1.4}$$

Since $\mathcal{T}$ is closed under quotients, we have $T_1 \in \mathcal{T}$. The short exact sequence (1.4) yields a triangle in $D^b(\mathcal{A})$

$$X^{-1} \to T_0 \to T_1 \xrightarrow{f} X^{-1}[1].$$

The composition $d_X^{-1}[1] \circ f$ is a morphism in $\mathrm{Hom}_{D^b(\mathcal{A})}(T_1, X^0[1]) = \mathrm{Ext}^1_{\mathcal{A}}(T_1, X^0)$, so it corresponds to an extension in $\mathcal{A}$

$$0 \to X^0 \to E \to T_1 \to 0.$$

In $D^b(\mathcal{A})$ we get a commutative diagram of triangles

$$\begin{tikzcd} X^0 \arrow[r, "\mu"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "g"] & T_1 \arrow[r] \arrow[d, "f"] & X^0[1] \arrow[d, equal] \\ X^0 \arrow[r, "u"] & X^\bullet \arrow[r, "v"] & X^{-1}[1] \arrow[r, "d_X^{-1}[1]"] & X^0[1], \end{tikzcd} \tag{1.5}$$

where $g \in \mathrm{Hom}_{D^b(\mathcal{A})}(E, X^\bullet)$ exists by definition of triangulated category. We need to show that $E \in \mathcal{Y}$ an that g is an epimorphism in $\mathcal{B}$.

In $\mathcal{A}$ there is a commutative diagram with exact horizontal arrows

$$\begin{tikzcd} 0 \arrow[r] & X^{-1} \arrow[r] \arrow[d, "d_X^{-1}"] & T_0 \arrow[r] \arrow[d, "e"] & T_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X^0 \arrow[r] & E \arrow[r] & T_1 \arrow[r] & 0. \end{tikzcd}$$

We have $\operatorname{coker} d_X^{-1} = H^0(X^\bullet) \in \mathcal{T}$ by hypothesis, $\operatorname{Im} e \in \mathcal{T}$ because $\mathcal{T}$ is closed under quotients. Moreover, by commutativity of the previous diagram, $\operatorname{coker} e \simeq \operatorname{coker} d_X^{-1} \in \mathcal{T}$. Therefore, E is an extension of objects in $\mathcal{T}$, so it belongs to $\mathcal{T} = \mathcal{Y}$ by Proposition 1.1.3.

We are left to prove that g is an epimorphism in $\mathcal{B}$. Let then $Y^\bullet \in \mathcal{B}$ and $h \in \operatorname{Hom}_{D^b(\mathcal{A})}(X^\bullet, Y^\bullet)$ be such that $hg = 0$. Consider the torsion exact sequence of $Y^\bullet$ in $\mathcal{B}$

$$0 \to t(Y^\bullet) \xrightarrow{\alpha} Y^\bullet \xrightarrow{\beta} Y^\bullet/t(Y^\bullet) \to 0,$$

where $t(Y^\bullet) \in \mathcal{X}$ and $Y^\bullet/t(Y^\bullet) \in \mathcal{Y}$. By the commutativity of (1.5) we get

$$hu = hg\mu = 0.$$

Therefore, there exists $h' \colon X^{-1}[1] \to Y^\bullet$ such that $h = h'v$. Hence

$$h'f\pi = h'vg = hg = 0.$$

Since π is an epimorphism in $\mathcal{B}$, this proves that $h'f = 0$. Moreover, since $Y^\bullet/t(Y^\bullet) \in \mathcal{Y} = \mathcal{T} \subseteq \mathcal{A}$, we get that

$$\beta h' \in \operatorname{Hom}_{D^b(\mathcal{A})}(X^{-1}[1], Y^\bullet/t(Y^\bullet)) = \operatorname{Hom}_{D^b(\mathcal{A})}(X^{-1}, Y^\bullet/t(Y^\bullet)[-1]) = 0.$$

Therefore, there exists $h'' \colon X^{-1}[1] \to t(Y^\bullet)$ such that $h' = \alpha h''$. Then

$$0 = h'f = \alpha h'' f.$$

since α is a monomorphism in $\mathcal{B}$, this proves that $h''f = 0$.

Applying the functor $\operatorname{Hom}_{D^b(\mathcal{A})}(-, t(Y^\bullet))$ to the triangle

$$T_0 \to T_1 \xrightarrow{f} X^{-1}[1] \to T_0[1],$$

we get an exact sequence

$$\operatorname{Hom}_{D^b(\mathcal{A})}(T_0[1], t(Y^\bullet)) \to \operatorname{Hom}_{D^b(\mathcal{A})}(X^{-1}[1], t(Y^\bullet)) \xrightarrow{-\circ f} \operatorname{Hom}(T_1, t(Y^\bullet)).$$

But $\operatorname{Hom}_{D^b(\mathcal{A})}(T_0[1], t(Y^\bullet)) = \operatorname{Hom}_{D^b(\mathcal{A})}(T_0, t(Y^\bullet)[-1]) = 0$, since $T_0 \in \mathcal{Y} = \mathcal{T}$ and $t(Y^\bullet)[-1] \in \mathcal{X}[-1] = \mathcal{F}$. Therefore

$$- \circ f \colon \operatorname{Hom}_{D^b(\mathcal{A})}(X^{-1}[1], t(Y^\bullet)) \to \operatorname{Hom}(T_1, t(Y^\bullet)).$$

is injective. Since $h'f = 0$, this proves that $h' = 0$ and hence that $h = 0$. This proves that g is an epimorphism. □

We now want to understand sufficient conditions for $D^b(\mathcal{B})$ and $D^b(\mathcal{A})$ to be equivalent. We want to construct a functor

$$G \colon D^b(\mathcal{B}) \to D^b(\mathcal{A})$$

and understand when it is an equivalence. Suppose that $\mathcal{T}$ is a tilting torsion class in $\mathcal{A}$, so that $\mathcal{Y}$ is cotilting torsion free in $\mathcal{B}$. We suppose moreover that the abelian category $\mathcal{B}$ has enough projective objects.

Let $\mathcal{P}_\mathcal{B} \subseteq \mathcal{B}$ be the subcategory formed by the projective objects is $\mathcal{B}$. Since $\mathcal{B}$ has enough projectives, there is a triangle equivalence

$$K^{-,b}(\mathcal{P}_\mathcal{B}) \xrightarrow{\simeq} D^b(\mathcal{B}).$$

Since $\mathcal{Y}$ is cotilting and $\mathcal{B}$ has enough projectives, there is an inclusion

$$\mathcal{P}_{\mathcal{B}} \subseteq \mathcal{Y} = \mathcal{T} \subseteq \mathcal{A}.$$

Therefore, we get an inclusion functor

$$K^{-,b}(\mathcal{P}_{\mathcal{B}}) \hookrightarrow K^{-,b}(\mathcal{A}).$$

On the other side, the natural functor $K(\mathcal{A}) \to D^b(\mathcal{A})$ restricts to a functor

$$Q\colon K^{-,b}(\mathcal{A}) \to D^b(\mathcal{A})$$

Composing, we get the desired functor

$$G\colon D^b(\mathcal{B}) \xrightarrow{\simeq} K^{-,b}(\mathcal{P}_{\mathcal{B}}) \hookrightarrow K^{-,b}(\mathcal{A}) \xrightarrow{Q} D^b(\mathcal{A}).$$

Notice that $G\colon D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is defined as a composition of exact functors, hence it is exact.

Dually, if instead of supposing $\mathcal{B}$ to have enough projectives we suppose $\mathcal{A}$ to have enough injectives, we can construct a functor

$$F\colon D^b(\mathcal{A}) \to D^b(\mathcal{B}).$$

Indeed, let $\mathcal{I}_{\mathcal{A}} \subseteq \mathcal{A}$ be the subcategory form by the injective objects.Then, there is a triangle equivalence

$$K^{+,b}(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\simeq} D^b(\mathcal{A}).$$

Since $\mathcal{T}$ is tilting and $\mathcal{A}$ has enough injectives, there is an inclusion

$$\mathcal{I}_{\mathcal{A}} \subseteq \mathcal{T} = \mathcal{Y} \subseteq \mathcal{B}.$$

Therefore, we get an inclusion functor

$$K^{+,b}(\mathcal{I}_{\mathcal{A}}) \hookrightarrow K^{+,b}(\mathcal{B}).$$

On the other side, the natural functor $K(\mathcal{B}) \to D^b(\mathcal{B})$ restricts to a functor

$$Q\colon K^{+,b}(\mathcal{B}) \to D^b(\mathcal{B})$$

Composing, we get the desired exact functor

$$F\colon D^b(\mathcal{A}) \xrightarrow{\simeq} K^{+,b}(\mathcal{I}_{\mathcal{A}}) \hookrightarrow K^{+,b}(\mathcal{B}) \xrightarrow{Q} D^b(\mathcal{B}).$$

Theorem 1.2.3. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in the abelian category $\mathcal{A}$ and suppose that $\mathcal{T}$ is a tilting torsion class in $\mathcal{A}$.

(a) If $\mathcal{B}$ has enough projectives, then the functor

$$G\colon D^b(\mathcal{A}) \to D^b(\mathcal{A})$$

is a triangle equivalence. Moreover, the restriction $G|_{\mathcal{B}}$ coincides with the identity functor $id_{\mathcal{B}}$.

(b) If $\mathcal{A}$ has enough injectives, then the functor

$$F \colon D^b(\mathcal{B}) \to D^b(\mathcal{B})$$

is a triangle equivalence. Moreover, the restriction $F|_{\mathcal{A}}$ coincides with the identity functor $id_{\mathcal{A}}$.

Proof. We prove (a), the proof of (b) is analogous. By [BBD82, Remark 3.1.17] it suffices to show that for all $B, B' \in \mathcal{B}$ the induced morphism

$$G_n \colon \operatorname{Hom}_{D^b(\mathcal{B})}(B, B'[n]) \to \operatorname{Hom}_{D^b(\mathcal{A})}(B, B'[n])$$

is bijective for all $n \in \mathbb{Z}$. Using [BBD82, Remark 3.1.17], we get that G_n bijective for $n \geq 1$ and that the bijectivity of G_n implies the injectivity of G_{n+1}.

Proceeding by induction, it then suffices to show that G_n is surjective for all n. Let $f \in \operatorname{Hom}_{D^b(\mathcal{A})}(B, B'[n])$, represented by the diagram

$$\begin{tikzcd} & X^\bullet & \\ B \arrow[ur, "\varphi"] & & B'[n], \arrow[ul, "s"'] \end{tikzcd}$$

where φ, s are morphisms in $K^b(\mathcal{A})$ and s is a quasi-isomorphism. Since $\mathcal{T}$ is tilting torsion, it is a cogenerator in $\mathcal{A}$. Then, by an argument analogous to the one present in [Huy06, Prop 2.35], we may find a complex $Y^\bullet \in D^b(\mathcal{A})$ with $Y^i \in \mathcal{T}$ and a quasi-isomorphism $t \colon X^\bullet \to Y^\bullet$.

Let $\widetilde{f} \in \operatorname{Hom}_{D^b(\mathcal{A})}$ be the morphism represented by the the following diagram

$$\begin{tikzcd} & Y^\bullet & \\ B \arrow[ur, "\widetilde{\varphi}"] & & B'[n], \arrow[ul, "\widetilde{s}"'] \end{tikzcd} \tag{1.6}$$

where $\widetilde{\varphi} = t\varphi$ and $\widetilde{s} = ts$. Since t is a quasi-isomorphism, f and $\widetilde{f}$ are actually the same morphism in $\operatorname{Hom}_{D^b(\mathcal{A})}(B, B'[n])$.

But the diagram (1.6) represents also a morphism $\widetilde{f}' \in \operatorname{Hom}_{D^b(\mathcal{A})}(B, B'[n])$. Clearly, $G_n(\widetilde{f}') = \widetilde{f} = f$. Hence, G_n is surjective and the proof is concluded. □

Notice that the triangle equivalence $G : D^b(\mathcal{B}) \to D^b(\mathcal{A})$ is defined in such a way that $G(\mathcal{X}) = \mathcal{F}[1]$ and $G(\mathcal{Y}) = \mathcal{T}$.

A natural question following the construction of $(\mathcal{B}; (\mathcal{X}, \mathcal{Y}))$ is what happens after tilting again, with respect to the torsion pair $(\mathcal{X}, \mathcal{Y})$. In particular, under the assumptions that give the triangle equivalence G, it seem natural to expect that tilting $\mathcal{B}$ with respect to $(\mathcal{X}, \mathcal{Y})$ should give once again $(A; (\mathcal{T}, \mathcal{F}))$. The following theorem formalizes this intuition.

Theorem 1.2.4. Suppose that either $\mathcal{B}$ has enough projectives or $\mathcal{A}$ has enough injectives, so that the functor G (resp. F) is a triangle equivalence between $D^b(\mathcal{B})$ and $D^b(\mathcal{A})$, according to Theorem 1.2.3. Then G (resp. F) induces an equivalence of categories $\Phi(\mathcal{B}; (\mathcal{X}, \mathcal{Y})) \simeq (\mathcal{A}; (\mathcal{T}, \mathcal{F}))$.

Proof. We suppose $\mathcal{B}$ to have enough projectives, the other case is analogous. Set

$$\Phi(\mathcal{B};(\mathcal{X},\mathcal{Y})) =: \left(\mathcal{A}';\left(\mathcal{T}',\mathcal{F}'\right)\right),$$

so that

$$\mathcal{A}' = \left\{X^\bullet \in D^b(\mathcal{B}) \mid H^i(X^\bullet) = 0 \text{ for all } i \neq 0,-1,\ H^0(X^\bullet) \in \mathcal{X},\ H^{-1}(X^\bullet) \in \mathcal{Y}\right\},$$
$$\mathcal{T}' = \mathcal{Y}[1], \quad \mathcal{F}' = \mathcal{X}.$$

We want to show that

$$G|_{\mathcal{A}'} \colon \mathcal{A}' \to \mathcal{A}[1] \subseteq D^b(\mathcal{A})$$

is an equivalence.

Let $X' \in \mathcal{A}'$. Then, in $D^b(\mathcal{B})$ there is a triangle

$$U \to X' \to V \to U[1],$$

with $U \in \mathcal{T}'$ and $V \in \mathcal{F}'$. Acting with the functor G, we get a triangle

$$G(U) \to G(X') \to G(V) \to G(U)[1]. \tag{1.7}$$

Notice that $G(\mathcal{T}') = G(\mathcal{Y}[1]) = \mathcal{T}[1]$ and $G(\mathcal{F}') = G(\mathcal{X}) = \mathcal{F}[1]$, so

$$H^i(G(U)) = H^i(G(V)) = 0 \text{ for all } i \neq -1.$$

Then, the long exact cohomology sequence obtained from (1.7) shows that $G(X')$ belongs to $\mathcal{A}[1]$.

Therefore,

$$G|_{\mathcal{A}'} \colon \mathcal{A}' \to \mathcal{A}[1] \subseteq D^b(\mathcal{A})$$

is a well defined fully faithful functor. It was already observed that $G(\mathcal{T}') = G(\mathcal{Y}[1]) = \mathcal{T}[1]$ and $G(\mathcal{F}') = G(\mathcal{X}) = \mathcal{F}[1]$, so to conclude that $G|_{\mathcal{A}'}$ is the desired equivalence it suffices to show that it is essentially surjective.

Let $X \in \mathcal{A}$ and consider its torsion exact sequence

$$0 \to t(X) \to X \to X/t(X) \to 0,$$

with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$. It gives rise to a triangle in $D^b(\mathcal{A})$

$$t(X)[1] \to X[1] \to X/t(X)[1] \xrightarrow{w} t(X)[2],$$

with $t(X)[1] \in \mathcal{T}[1]$ and $X/t(X)[1] \in \mathcal{F}[1]$. Therefore, there exist $U \in \mathcal{Y}[1] = \mathcal{T}'$ and $V \in \mathcal{X} = \mathcal{F}'$ such that

$$G(U) = t(X)[1], \quad G(V) = X/t(X)[1].$$

By full faithfulness of G, there exists $w' \in \operatorname{Hom}_{D^b(\mathcal{B})}(V, U[1])$ such that $G(w') = w$. The morphism w' may be completed to a triangle in $D^b(\mathcal{B})$

$$U \to X' \to V \xrightarrow{w'} U[1]. \tag{1.8}$$

By looking at the long exact cohomology sequence of (1.8), we see that X' belongs to $\mathcal{A}'$. Applying the functor G to (1.8), we get the following commutative diagram

$$\begin{tikzcd}
G(U) \arrow[r] \arrow[d, equal] & G(X') \arrow[r] \arrow[d, "h"] & G(V) \arrow[r] \arrow[d, equal] & G(U)[1] \arrow[d, equal] \\
t(X)[1] \arrow[r] & X[1] \arrow[r] & X/t(X)[1] \arrow[r] & t(X)[2].
\end{tikzcd}$$

Therefore, $X[1]$ is isomorphic to $G(X')$ via the morphism h. This proves that

$$G|_{\mathcal{A}'} \colon \mathcal{A}' \to \mathcal{A}[1] \subseteq D^b(\mathcal{A})$$

is essentially surjective, thus concluding the proof. $\square$

Chapter 2

Perverse coherent sheaves

2.1 Preliminaries

The content of this chapter is based on [Van04].

Let $f\colon Y \to X$ be a morphism between Noetherian equidimensional schemes. Suppose that the following hypotheses are satisfied:

1. The morphism f is projective and birational.
2. There is a point $p \in X$ such that $f^{-1}(p) =: C$ is a curve contained in Y. Moreover
$$f|_{Y \smallsetminus C}\colon Y \smallsetminus C \to X \smallsetminus \{p\}$$
is an isomorphism.
3. $\mathbf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$.

Lemma 2.1.1. There is a natural isomorphism $\mathbf{R}f_*\mathbf{L}f^* \simeq \mathrm{id}$.

Proof. The projection formula [BBH09, §A.83] shows that if $\mathcal{E}$ is a quasi-coherent sheaf of $\mathcal{O}_X-$modules, then there is a natural isomorphism
$$\mathbf{R}f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{E} \simeq \mathbf{R}f_*(\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathbf{L}f^*\mathcal{E}).$$
By using the hypothesis $\mathbf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$, we get
$$\mathcal{E} \simeq \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E} \simeq \mathbf{R}f_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{E}.$$
While on the other hand, clearly
$$\mathbf{R}f_*(\mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathbf{L}f^*\mathcal{E}) \simeq \mathbf{R}f_*(\mathbf{L}f^*\mathcal{E}).$$
So in conclusion, for a quasi-coherent sheaf $\mathcal{E}$
$$\mathcal{E} \simeq \mathbf{R}f_*\mathbf{L}f^*\mathcal{E},$$
proving the thesis. □

Lemma 2.1.2. For any coherent sheaf $\mathcal{E} \in \mathrm{coh}(Y)$, we have

1. $\mathbf{R}^k f_*\mathcal{E} = 0$ for all $k \geq 2$;

2. $\mathbf{R}^1 f_* \mathcal{E}$ is supported at p.

Proof. The first statement is just [Har13, §III, Corollary 11.2], using the fact that the fibers of f have either dimension 0 or 1.

To prove the second statement, let $x \in X \setminus \{p\}$ and observe that, by [Har13, §III, Proposition 8.1], we have

$$(\mathbf{R}^1 f_* \mathcal{E})_x = \lim_{V \ni x} H^1(f^{-1}(V), \mathcal{E}|_{f^{-1}(V)}) = \lim_{U \ni f^{-1}(x)} H^1(U, \mathcal{E}|_U).$$

To conclude, it is enough to take an affine neighborhood of the point $y = f^{-1}(x)$. □

Corollary 2.1.3. For any coherent sheaf $\mathcal{F} \in \mathrm{coh}(X)$, we have $\mathbf{R}^1 f_* f^* F = 0$.

Proof. Notice that $\mathbf{L} f^* \mathcal{F}$ is concentrated in negative degrees. Indeed, the functor $\mathbf{L} f^*$ maps $D^-(\mathrm{coh}(X))$ to $D^-(\mathrm{coh}(Y))$. By seeing $f^* \mathcal{F}$ as a complex concentrated in degree zero, there is a canonical morphism

$$\mathbf{L} f^* \mathcal{F} \to f^* \mathcal{F}.$$

This morphism induces isomorphisms in cohomology in degrees ≥ 0. Therefore, it fits into an exact triangle

$$G \to \mathbf{L} f^* \mathcal{F} \to f^* \mathcal{F},$$

with G concentrated in degrees ≤ -1. Applying derived pushforward, we get a triangle

$$\mathbf{R} f_* G \to \mathbf{R} f_* \mathbf{L} f^* \mathcal{F} \to \mathbf{R} f_* f^* \mathcal{F}.$$

By Lemma 2.1.1, the middle term is actually isomorphic to $\mathcal{F}$, so as a complex it is concentrated in degree 0. Passing to the long exact sequence in cohomology, we obtain

$$0 \to \mathbf{R}^1 f_* f^* \mathcal{F} \to \mathbf{R}^2 f_* G \to 0.$$

Consider the spectral sequence

$$E_2^{p,q} := \mathbf{R}^p f_* \mathbf{H}^q(G) \Longrightarrow \mathbf{R}^{p+q} f_*(G).$$

The left hand side vanishes fo $p \geq 2$ or $q \geq 0$, so $\mathbf{R}^2 f_* G$ vanishes as well. Therefore

$$\mathbf{R}^1 f_* f^* \mathcal{F} = 0,$$

proving the claim. □

2.2 Definition of perverse coherent sheaves

In the following, we are going to study two categories of perverse coherent sheaves induced on Y by the morphism f.

Let $\mathcal{C}$ be defined as

$$\mathcal{C} = \{\mathcal{E} \in \mathrm{coh}(Y) \mid \mathbf{R} f_* \mathcal{E} = 0\}.$$

The following lemma is [Bri02, Lemma 3.1].

Lemma 2.2.1. For $F \in D^b(\mathrm{coh}(Y))$, one has $\mathbf{R} f_* F = 0$ if and only if $\mathbf{H}^i(F) \in \mathcal{C}$ for all i.

We now define two torsion pairs on $D^b(\mathrm{coh}(Y))$

$$\begin{aligned}
\mathcal{T}_{-1} &= \{T \in \mathrm{coh}(Y) \mid \mathbf{R}^1 f_*(T) = 0,\ \mathrm{Hom}(T, \mathcal{C}) = 0\}, \\
\mathcal{F}_{-1} &= \{F \in \mathrm{coh}(Y) \mid f_*(F) = 0\}, \\
\mathcal{T}_0 &= \{T \in \mathrm{coh}(Y) \mid \mathbf{R}^1 f_*(T) = 0\}, \\
\mathcal{F}_0 &= \{F \in \mathrm{coh}(Y) \mid f_*(F) = 0,\ \mathrm{Hom}(\mathcal{C}, F) = 0\}.
\end{aligned}$$

In order to show that $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$ and $(\mathcal{T}_0, \mathcal{F}_0)$ are a torsion pairs, we first observe some elementary properties.

Lemma 2.2.2. The following properties hold for $p = 0, 1$.

(a) $\mathcal{T}_p$ is closed under quotients and under extensions.

(b) $\mathcal{F}_p$ is closed under subobjects.

(c) $\mathcal{T}_p \cap \mathcal{F}_p = \{0\}$.

(d) $f^* f_* E$ belongs to $\mathcal{T}_{-1}$ for all $E \in \mathrm{coh}(Y)$.

Proof. Let $T \in \mathcal{T}_p$ and let T' be a quotient object of T. By looking at the long exact sequence obtained by derived pushforward from the short exact sequence

$$0 \to K \to T \to T' \to 0$$

and using the fact that $\mathbf{R}^2 f_* K = 0$, we get that $\mathbf{R}^1 f_* T' = 0$. This concludes the proof that $\mathcal{T}_0$ is closed under quotients. For $p = 1$, suppose that there is a nonzero morphism $T' \to E$ for some $E \in \mathcal{C}$. Then the composition

$$T \twoheadrightarrow T' \to E$$

yields a nonzero morphism in $\mathrm{Hom}(T, E)$. This is absurd, proving that $T' \in \mathcal{T}_{-1}$ and hence that $\mathcal{T}_{-1}$ is closed under quotients.
Let us consider an extension

$$0 \to T_0 \xrightarrow{i} T \xrightarrow{\pi} T_1 \to 0$$

with $T_0, T_1 \in \mathcal{T}_p$. We want to show that $T \in \mathcal{T}_p$. By looking at the long exact sequence of derived pushforward, we get that $\mathbf{R}^1 f_* T = 0$, immediately concluding the case $p = 0$. Suppose then $p = 1$, consider a sheaf $\mathcal{E} \in \mathcal{C}$ and let $\varphi \in \mathrm{Hom}(T, \mathcal{E})$ be a morphism. Composing with the inclusion i we get a morphism in $\mathrm{Hom}(T_0, \mathcal{E})$, which must be zero. Therefore,

$$\varphi(\mathrm{Im}\ i) = \varphi(\mathrm{Ker}\ \pi) = 0.$$

Since $T_1 \simeq T / \mathrm{Ker}\ \pi$, the induced morphism

$$\tilde{\varphi} \colon T / \mathrm{Ker}\ \pi \to \mathcal{E}$$

must be zero. This proves that φ is the zero morphism and hence that $\mathcal{T}_{-1}$ is closed under extensions. This concludes the proof of (a).

Let $F \in \mathcal{F}_p$ and let F' be a subsheaf of F. Since f_* is left exact, $f_* F'$ is a subsheaf of $f_* F = 0$ so it must be zero, concluding the case $p = 1$. For $p = 0$, notice that a nonzero morphism $E \to F'$ for $E \in \mathcal{C}$ would yield a nonzero morphism $E \to F$. In conclusion, both $\mathcal{F}_{-1}$ and $\mathcal{F}_0$ are closed under subobjects, proving (b).

Let $E \in \mathcal{T}_p \cap \mathcal{F}_p$. Then $\mathbf{R}^i f_* E = 0$ for all $i \geq 0$, so E belongs to $\mathcal{C}$. But since E belongs either to $\mathcal{T}_{-1}$ or to $\mathcal{F}_0$ as well, this implies that $\mathrm{Hom}(E, E) = 0$ and hence that $E = 0$. This proves (c).

We are left to prove (d). Using Corollary 2.1.3, we get that

$$\mathbf{R}^1 f_*(f^* f_* E) = 0.$$

Moreover, for $\mathcal{E} \in \mathcal{C}$ we have

$$\mathrm{Hom}(f^* f_* E, \mathcal{E}) = \mathrm{Hom}(f_* E, f_* \mathcal{E}) = \mathrm{Hom}(f_* E, 0) = 0.$$

This proves that $f^* f_* E$ belongs to $\mathcal{T}_{-1}$. □

Proposition 2.2.3. The pairs $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$, $(\mathcal{T}_0, \mathcal{F}_0)$ are torsion pairs on $\mathrm{coh}(Y)$.

Proof. Let us first prove that $\mathrm{Hom}(\mathcal{T}_p, \mathcal{F}_p)$ is zero. Let $\varphi \colon T \to F$ be a morphism, with $T \in \mathcal{T}_p$ and $F \in \mathcal{F}_p$. Then the image of φ is both a subobject of F and a quotient of T. So $\mathrm{Im}\ \varphi \in \mathcal{T}_p \cap \mathcal{F}_p$ is zero and φ is the zero morphism.

Let E be a coherent sheaf on Y. We want to find a short exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}_p$ and $F \in \mathcal{F}_p$. Since $\mathcal{T}_p$ is closed under extensions (and in particular under finite direct sums), then E contains a maximal subsheaf T in $\mathcal{T}_p$. Notice that by maximality of T we have $\mathrm{Hom}(\mathcal{T}_p, E/T) = 0$. Indeed, if $\varphi \colon T' \to E/T$ is a nonzero morphism, then the image of φ is a nontrivial subsheaf of E/T that belongs to $\mathcal{T}_p$, contradicting the maximality of T.
In order to conclude, it is enough to show that $F := E/T$ belongs to $\mathcal{F}_p$. By property (d) of Lemma 2.2.2, $f^* f_* F \in \mathcal{T}_{-1}$. Then

$$\mathrm{Hom}(f_* F, f_* F) = \mathrm{Hom}(f^* f_* F, F) = 0.$$

This proves that $f_* F = 0$ and concludes the case $p = -1$. To conclude the case $p = 0$, we need also to prove that $\mathrm{Hom}(\mathcal{C}, F) = 0$. But in this case $\mathcal{C} \subseteq \mathcal{T}_0$ and we already noticed that $\mathrm{Hom}(\mathcal{T}_0, F) = 0$. □

These torsion pairs induces the following t-structures on $D^b(\mathrm{coh}(Y))$.

$$D^b(\mathrm{coh}(Y))^{\leq 0, p} = \{E \in D^b(\mathrm{coh}(Y)) \mid \mathbf{H}^i(E) = 0 \text{ for } i > 0 \text{ and } \mathbf{H}^0(E) \in \mathcal{T}_p\},$$
$$D^b(\mathrm{coh}(Y))^{\geq 0, p} = \{E \in D^b(\mathrm{coh}(Y)) \mid \mathbf{H}^i(E) = 0 \text{ for } i < -1 \text{ and } \mathbf{H}^{-1}(E) \in \mathcal{F}_p\}.$$

The heart of the t-structure is

$$\mathrm{Per}^p(Y/X) = \{E \in D^b(\mathrm{coh}(Y)) \mid \mathbf{H}^i(E) = 0 \text{ for } i \neq 0, -1, \\ \mathbf{H}^{-1}(E) \in \mathcal{F}_p,\ \mathbf{H}^0(E) \in \mathcal{T}_p\},$$

for $p = 0, 1$.

Since $\mathrm{Per}^p(Y/X)$ is the heart of a t-structure on $D^b(\mathrm{coh}(Y))$, we know that it is an abelian category.
Notice that the sheaves in $\mathcal{F}_p$ have low-dimensional support.

Lemma 2.2.4. If $F \in \mathrm{coh}(Y)$ belongs $\mathcal{F}_p$, then F is supported on the curve C, so in particular $\dim F \leq 1$.

Proof. Since f is an isomorphism outside of C, we have

$$(f_*F)_x = F_{f^{-1}(x)}$$

for all $x \in X \setminus \{p\}$. By hypothesis $(f_*F)_x = 0$ for all $x \in X$. So in conclusion

$$F_y = 0$$

for all $y \in Y \setminus C$. Therefore, the support of F is contained in C and has dimension at most one. □

Lemma 2.2.5. The structure sheaf $\mathcal{O}_Y$ belongs to $\mathcal{T}_{-1}$. As a consequence, any coherent sheaf on Y generated by global sections is in $\mathcal{T}_{-1}$.

Proof. Since $\mathbf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$,

$$f_*\mathcal{O}_Y = \mathbf{R}^0 f_*\mathcal{O}_Y = \mathbf{H}^0(\mathcal{O}_X) = \mathcal{O}_X \quad \text{and} \quad \mathbf{R}^i f_*\mathcal{O}_Y = \mathbf{H}^i(\mathcal{O}_X) = 0 \text{ for } i > 0.$$

This proves that $\mathbf{R}^1 f_*\mathcal{O}_Y = 0$ and that $f^*f_*\mathcal{O}_Y = f^*\mathcal{O}_X = \mathcal{O}_Y$. Therefore, for $\mathcal{E} \in \mathcal{C}$ we have

$$\mathrm{Hom}(\mathcal{O}_Y, \mathcal{E}) = \mathrm{Hom}(f^*f_*\mathcal{O}_Y, \mathcal{E}) = \mathrm{Hom}(f_*\mathcal{O}_Y, f_*\mathcal{E}) = \mathrm{Hom}(f_*\mathcal{O}_Y, 0) = 0,$$

proving that $\mathcal{O}_Y \in \mathcal{T}_{-1}$.
Since $\mathcal{T}_{-1}$ is closed under extensions, clearly $\mathcal{O}_Y^{\oplus c} \in \mathcal{T}_{-1}$ for all c. If E is a globally generated sheaf, it is a quotient of $\mathcal{O}_Y^{\oplus c}$ for some c, therefore $E \in \mathcal{T}_{-1}$. □

Remark 2.2.6. It follows straightforwardly that the previous statements hold true for $\mathcal{T}_0$ as well.

We now aim to understand better the structure of $\mathrm{Per}^p(Y/X)$, in particular studying the projective objects. In order to do this, it is necessary to give a different equivalent condition for a coherent sheaf to lie in $\mathcal{T}_{-1}$ or in $\mathcal{F}_0$.

Lemma 2.2.7. Let $\mathcal{E} \in \mathrm{coh}(Y)$ and consider the natural morphism $f^*f_*\mathcal{E} \to \mathcal{E}$, induced by the adjunction $f^* \dashv f_*$. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ be image subsheaf. Then $\mathbf{R}^1 f_*\mathcal{E}_0 = 0$.

Proof. Under the above assumptions, let $\mathcal{F} := f_*\mathcal{E}$. Then, we have a short exact sequence

$$0 \to K \to f^*\mathcal{F} \to \mathcal{E}_0 \to 0.$$

By taking the long exact sequence obtained by derived pushforward, we get the exact sequence

$$\mathbf{R}^1 f_*K \to \mathbf{R}^1 f_*f^*\mathcal{F} \to \mathbf{R}^1 f_*\mathcal{E}_0 \to 0.$$

By Corollary 2.1.3 the middle term vanishes, thus proving $\mathbf{R}^1 f_*\mathcal{E}_0 = 0$. □

Lemma 2.2.8. Let $T \in \mathrm{coh}(Y)$ and consider the natural morphism $f^*f_*T \to T$, induced by the adjunction $f^* \dashv f_*$. Let $T_0 \subseteq T$ be image subsheaf. If $\mathbf{R}^1 f_*T = 0$, then T/T_0 belongs to $\mathcal{C}$.

Proof. We need to show that $\mathbf{R}^i f_*(T/T_0) = 0$ for all $i \geq 0$. The only non trivial cases are $i = 0, 1$.

For the case $i = 1$, consider the short exact sequence of sheaves

$$0 \to T_0 \to T \to T/T_0 \to 0.$$

The long exact sequence obtained by taking derived pushforward of f gives the exact sequence

$$\mathbf{R}^1 f_*(T) \to \mathbf{R}^1 f_*(T/T_0) \to \mathbf{R}^2 f_*(T_0).$$

But $\mathbf{R}^1 f_*(T) = 0$ by hypothesis and $\mathbf{R}^2 f_*(T_0) = 0$, therefore $\mathbf{R}^1 f_*(T/T_0) = 0$.

For the case $i = 0$, we have to show that $f_*(T/T_0) = 0$. Suppose then that $f_*(T/T_0) \neq 0$. Observe that this implies that the morphism $f^* f_*(T/T_0) \to T/T_0$ is not the zero map. Indeed, acting via f_* we get a morphism $f_* f^* f_*(T/T_0) \to f_*(T/T_0)$. But by Lemma 2.1.1, we see that

$$f_* f^* f_*(T/T_0) = \mathbf{R}^0 f_*(f^* f_*(T/T_0)) = \mathbf{R}^0 f_* \mathbf{L}^0 f^*(f_*(T/T_0)) \simeq f_*(T/T_0).$$

Moreover, by naturality, the induced morphism $f_*(T/T_0) \to f_*(T/T_0)$ must be the identity map. Since we supposed $f_*(T/T_0) \neq 0$, the identity map is not the zero map. In conclusion, by acting via f_* and composing with an isomorphism we obtained a nonzero morphism, so the considered morphism $f^* f_*(T/T_0) \to T/T_0$ is not zero.

By the previous Lemma 2.2.7, $\mathbf{R}^1 f_*(T_0) = 0$. This implies that $f_*(T) \to f_*(T/T_0)$ is surjective. So acting via pullback we get that $f^* f_*(T) \to f^* f_*(T/T_0)$ is surjective. Take now the composition

$$\Phi\colon f^* f_*(T) \to f^* f_*(T/T_0) \to T/T_0.$$

Since the first map is surjective and the second is nonzero, Φ is nonzero. On the other hand, by naturality of the adjunction $f^* \dashv f_*$, Φ can be obtained also as

$$f^* f_*(T) \to T \to T/T_0,$$

which is zero by construction. This is absurd, proving $f_*(T/T_0) = 0$ and hence the thesis. □

This lemma allows to characterize the objects in $\mathcal{T}_{-1}$.

Proposition 2.2.9. Let T be a coherent sheaf on Y. Then T belongs to $\mathcal{T}_{-1}$ if and only if the natural morphism $f^* f_* T \to T$ is surjective.

Proof. Suppose $T \in \mathcal{T}_{-1}$ and let $T_0 \subseteq T$ be the image of the morphism. Since $\mathbf{R}^1 f_* T = 0$, we can apply the previous lemma and get that $T/T_0 \in \mathcal{C}$.
But then the projection morphism

$$T \to T/T_0$$

must be zero. So $T = T_0$, i.e. the morphism $f^* f_* T \to T$ is surjective.

Suppose now that the morphism is surjective. We prove that T satisfies the two conditions required in order for a coherent sheaf to be in $\mathcal{T}_{-1}$.
As in the previous lemma, consider the short exact sequence

$$0 \to K \to f^* f_* T \to T \to 0$$

and take the associated long exact sequence via derived pushforward. As before, $\mathbf{R}^1 f_*(f^* f_* T) = 0$ by Corollary 2.1.3 and $\mathbf{R}^2 f_* K = 0$, giving $\mathbf{R}^1 f_* T = 0$ as required.
Let now $\mathcal{E}$ be in $\mathcal{C}$, we show that $\mathrm{Hom}(T, \mathcal{E}) = 0$. Applying the functor $\mathrm{Hom}(\_, \mathcal{E})$ to the same short exact sequence considered earlier, we get the exact sequence

$$0 \to \mathrm{Hom}(T, \mathcal{E}) \to \mathrm{Hom}(f^* f_* T, \mathcal{E}).$$

Notice that $\mathrm{Hom}(f^* f_* T, \mathcal{E}) = \mathrm{Hom}(f_* T, f_* \mathcal{E})$ is zero, since $f_* \mathcal{E} = 0$. So, $\mathrm{Hom}(T, \mathcal{E}) = 0$ as required. $\square$

Although not needed in what follows, it's worth mentioning that there is a dual statement to Lemma 2.2.9, characterizing the objects in $\mathcal{F}_0$.

If E is a coherent sheaf on Y, the composition

$$E \to f^! \mathbf{R} f_* E \to f^!((\mathbf{R}^1 f_* E)[-1])$$

yields a canonical morphism

$$\phi_E \colon E \to \mathbf{H}^{-1}(f^! \mathbf{R}^1 f_* E).$$

Lemma 2.2.10 ([Van04, Lemma 3.1.5])**.** Let F be a coherent sheaf on Y. Then F belongs to $\mathcal{F}_0$ if and only if the canonical morphism ϕ_F is injective.

2.3 Affine base

We now study the case where the base $X = \operatorname{Spec} R$ is affine. Under this additional hypothesis, we can give an explicit description of the sheaf $f^* f_* T$ and of the canonical morphism $f^* f_* T \to T$.

Notice that $f_* \mathcal{O}_Y = \mathcal{O}_X$ yields

$$H^0(Y, \mathcal{O}_Y) = H^0(X, f_* \mathcal{O}_Y) = H^0(X, \mathcal{O}_X) = R.$$

Lemma 2.3.1. There is an isomorphism $f^* f_* T \simeq H^0(Y, T) \otimes_R \mathcal{O}_Y$, where $H^0(Y, T)$ is seen as a constant sheaf on Y. Under this isomorphism, the natural morphism $f^* f_* T \to T$ induces the natural morphism $H^0(Y, T) \otimes_R \mathcal{O}_Y \to T$.

Proof. Since X is affine and $f_* T$ is quasi-coherent on X

$$f_* T = H^0 \widetilde{(X, f_* T)} = H^0 \widetilde{(Y, T)}.$$

Now, we use [Liu02, §5.1, Proposition 1.14(b)] in order to compute the sections of $f^* f_* T$ on an affine open $U \subseteq Y$:

$$f^*(f_* T)(U) = f_* T(X) \otimes_R \mathcal{O}_Y(U).$$

Therefore

$$f^* f_* T(U) = H^0(Y, T) \otimes_R \mathcal{O}_Y(U).$$

This proves the required isomorphism. The second statement follows by naturality. $\square$

Combining Lemma 2.3.1 with Proposition 2.2.9, we get a further characterization of $\mathcal{T}_{-1}$ in this case.

Lemma 2.3.2. If $X = \operatorname{Spec} R$ is affine, then a coherent sheaf $T \in \mathrm{coh}(Y)$ belongs to $\mathcal{T}_{-1}$ if and only if it is generated by global sections.

The following property will prove useful in the following computations.

Lemma 2.3.3. Let $f\colon Y \to X$ be a projective morphism between Noetherian schemes with fibers of dimension at most n. Suppose that X is affine. If $\mathcal{N}$ is a coherent sheaf on Y, then $H^i(Y, \mathcal{N}) = 0$ for all $i > n$. Similarly, if $\mathcal{M}$, $\mathcal{E}$ are coherent sheaves on Y, with $\mathcal{M}$ locally free, then $\mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{E}) = 0$ for all $i > n$.

Proof. It suffices to prove the first statement, since $\mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{E}) = H^i(Y, \mathcal{M}^\vee \otimes_{\mathcal{O}_Y} \mathcal{E})$. We use the Leray spectral sequence

$$E_2^{p,q} = H^p(X, \mathbf{R}^q f_* \mathcal{N}) \Longrightarrow H^{p+q}(Y, \mathcal{N}). \tag{2.1}$$

Notice that, since X is affine, $H^p(X, \mathbf{R}^q f_* \mathcal{N}) = 0$ for all $p > 0$. Moreover, $\mathbf{R}^q f_* \mathcal{N}$ vanishes for $q > n$. Hence, $E_2^{p,q} = 0$ for all $p > 0$ or $q > n$, proving the claim. □

Remark 2.3.4. In the case under study, we get $H^i(Y, \mathcal{N}) = 0$ for all $i > 1$. Moreover, if $\mathbf{R}^1 f_* \mathcal{N} = 0$, e.g. if $\mathcal{N} \in \mathcal{T}_p$, using Formula (2.1) we get that $H^1(Y, \mathcal{N})$ vanishes too. In particular, $H^1(Y, \mathcal{O}_Y) = 0$.

Definition 2.3.5. Let $\mathfrak{V}$ be the category of vector bundles $\mathcal{M}$ on Y (i.e., locally free sheaves) that are generated by global sections and such that $H^1(Y, \mathcal{M}^\vee) = 0$; and let $\mathfrak{V}^\vee := \{\mathcal{M}^\vee \mid \mathcal{M} \in \mathfrak{V}\}$.

Remark 2.3.6. The category $\mathfrak{V}$ is closed under direct sums and direct summands.

Notice that, by the Lemma 2.3.2, the objects of $\mathfrak{V}$ belong to $\mathcal{T}_{-1}$ as well. So they are (as complexes concentrated in degree zero) also objects in $\mathrm{Per}^{-1}(Y/X)$. Similarly, the Leray spectral sequence (2.1) shows that the objects in $\mathfrak{V}^\vee$ belong to $\mathcal{T}_0$ and hence to $\mathrm{Per}^0(Y/X)$.

In the next proposition, we will compute $\mathrm{Ext}^i_Y(\mathcal{M}, E)$ for $\mathcal{M} \in \mathfrak{V}$ and $E \in \mathrm{Per}^{-1}(Y/X)$. As usually, by $\mathrm{Ext}^i_Y(\mathcal{M}, E)$ we mean

$$\mathrm{Ext}^i_Y(\mathcal{M}, E) = \mathbf{H}^i(\mathbf{R}\,\mathrm{Hom}_Y(\mathcal{M}, E)) = \mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathcal{M}, E[i]).$$

This does not a priori coincide with Ext computed in the abelian category $\mathrm{Per}^{-1}(Y/X)$, but what we will prove is that the vanishing of the former implies the vanishing of the latter.

Proposition 2.3.7. If $\mathcal{M} \in \mathfrak{V}$, then for all $E \in \mathrm{Per}^{-1}(Y/X)$ and for all $i > 0$ we have $\mathrm{Ext}^i_Y(\mathcal{M}, E) = 0$. Therefore, $\mathcal{M}$ is a projective object in $\mathrm{Per}^{-1}(Y/X)$.

Proof. Since $\mathrm{Per}^{-1}(Y/X)$ is obtained as the heart of the t-structure induced by the torsion pair $(\mathcal{T}_{-1}, \mathcal{F}_{-1})$, every $E \in \mathrm{Per}^{-1}(Y/X)$ lies in a triangle

$$\mathbf{H}^{-1}(E)[1] \to E \to \mathbf{H}^0(E),$$

with $\mathbf{H}^{-1}(E) \in \mathcal{F}_{-1}$ and $\mathbf{H}^0(E) \in \mathcal{T}_{-1}$.
Therefore, by taking the long exact Ext sequence, it suffices to show that

$$\mathrm{Ext}^i_Y(\mathcal{M}, F[1]) = 0 \quad \text{and} \quad \mathrm{Ext}^i_Y(\mathcal{M}, T) = 0$$

for all $i > 0$ and for all $F \in \mathcal{F}_{-1}$ and $T \in \mathcal{T}_{-1}$.
Notice that

$$\mathrm{Ext}^i_Y(\mathcal{M}, F[1]) = \mathrm{Ext}^{i+1}_Y(\mathcal{M}, F).$$

Conclude using Lemma 2.3.3 that $\mathrm{Ext}^i_Y(\mathcal{M}, F[1]) = 0$ for $i > 0$.
By Lemma 2.3.2, T is generated by global sections. So there is a surjective morphism

$$\mathcal{O}_Y^{\oplus l} \twoheadrightarrow T.$$

The long exact Ext sequence obtained by $0 \to K \to \mathcal{O}_Y^{\oplus l} \to T \to 0$ gives the exact sequence

$$\mathrm{Ext}^1_Y(\mathcal{M}, \mathcal{O}_Y^{\oplus l}) \to \mathrm{Ext}^1_Y(\mathcal{M}, T) \to \mathrm{Ext}^2_Y(\mathcal{M}, K).$$

Notice that

$$\mathrm{Ext}^1_Y(\mathcal{M}, \mathcal{O}_Y^{\oplus l}) = H^1(Y, \mathcal{M}^\vee \otimes \mathcal{O}_Y^{\oplus l}) = H^1(Y, (\mathcal{M}^\vee)^{\oplus l}) = H^1(Y, \mathcal{M}^\vee)^{\oplus l} = 0,$$

and $\mathrm{Ext}^2_Y(\mathcal{M}, K) = 0$ by Lemma 2.3.3, proving that $\mathrm{Ext}^1_Y(\mathcal{M}, T) = 0$. The vanishing of $\mathrm{Ext}^i_Y(\mathcal{M}, T)$ for $i > 1$ follows again from Lemma 2.3.3, thus proving the first statement.

We now prove that the functor $\mathrm{Hom}_{\mathrm{Per}^{-1}(Y/X)}(\mathcal{M}, -)$ is exact, i.e. $\mathcal{M}$ is projective in $\mathrm{Per}^{-1}(Y/X)$.

First, notice that by definition of $\mathrm{Per}^{-1}(Y/X)$ we have

$$\mathrm{Hom}_{\mathrm{Per}^{-1}(Y/X)}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathcal{M}, \mathcal{N}) = \mathrm{Ext}^0_Y(\mathcal{M}, \mathcal{N}) \tag{2.2}$$

for any object $\mathcal{N} \in \mathrm{Per}^{-1}(Y/X)$.

Now, consider a short exact sequence

$$0 \to E \to F \to G \to 0$$

in $\mathrm{Per}^{-1}(Y/X)$. It corresponds to a distinguished triangle

$$E \to F \to G \to E[1]$$

in $D^b(\mathrm{coh}(Y))$. Therefore, by applying $\mathbf{R}\,\mathrm{Hom}_Y(\mathcal{M}, -)$ and taking cohomology, we get the long exact sequence

$$\mathrm{Ext}^i(\mathcal{M}, E) \to \mathrm{Ext}^i(\mathcal{M}, F) \to \mathrm{Ext}^i(\mathcal{M}, G) \to \mathrm{Ext}^{i+1}(\mathcal{M}, E).$$

By the first statement of this proposition, $\mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{N}) = 0$ for $i > 0$ and any $\mathcal{N} \in \mathrm{Per}^{-1}(Y/X)$. So actually the above sequence reduces to the short exact sequence

$$0 \to \mathrm{Ext}^0_Y(\mathcal{M}, E) \to \mathrm{Ext}^0_Y(\mathcal{M}, F) \to \mathrm{Ext}^0_Y(\mathcal{M}, G) \to 0.$$

Then the observation (2.2) proves that $\mathrm{Hom}_{\mathrm{Per}^{-1}(Y/X)}(\mathcal{M}, -)$ is an exact functor, i.e. $\mathcal{M}$ is a projective object in $\mathrm{Per}^{-1}(Y/X)$. □

An analogous statement holds true for $\mathrm{Per}^0(Y/X)$.

Proposition 2.3.8. If $\mathcal{N} \in \mathfrak{V}^\vee$, then for all $E \in \mathrm{Per}^0(Y/X)$ and for all $i > 0$ we have $\mathrm{Ext}^i_Y(\mathcal{N}, E) = 0$. Therefore, $\mathcal{N}$ is a projective object in $\mathrm{Per}^0(Y/X)$.

Proof. As in Proposition 2.3.7, it suffices to show that

$$\mathrm{Ext}^i_Y(\mathcal{N}, F[1]) = 0 \quad \text{and} \quad \mathrm{Ext}^i_Y(\mathcal{N}, T) = 0$$

for all $i > 0$ and for all $F \in \mathcal{F}_0$ and $T \in \mathcal{T}_0$.
By Lemma 2.3.3,

$$\mathrm{Ext}^i_Y(\mathcal{N}, F[1]) = \mathrm{Ext}^{i+1}_Y(\mathcal{N}, F) = 0$$

for all $i > 0$ and
$$\mathrm{Ext}^i_Y(\mathcal{N}, T) = 0$$
for all $i > 1$. It remains to show the vanishing of $\mathrm{Ext}^1_Y(\mathcal{N}, T) = H^1(Y, \mathcal{N}^\vee \otimes_{\mathcal{O}_Y} T)$. Since $\mathcal{N} = \mathcal{M}^\vee$ for $\mathcal{M} \in \mathfrak{V}$ and $\mathcal{M}$ is a vector bundle, there is a canonical isomorphism $\mathcal{N}^\vee = \mathcal{M}$. By definition of $\mathfrak{V}$, $\mathcal{M}$ is generated by global sections, so there is a surjective morphism
$$\mathcal{O}_Y^{\oplus l} \twoheadrightarrow \mathcal{M}.$$
Tensoring with T we get an exact sequence
$$0 \to K \to T^{\oplus l} \to \mathcal{M} \otimes_{\mathcal{O}_Y} T \to 0.$$
As observed in Remark 2.3.4, both $H^1(Y, T)$ and $H^2(Y, K)$ vanish, therefore
$$\mathrm{Ext}^1_Y(\mathcal{N}, T) = H^1(Y, \mathcal{M} \otimes_{\mathcal{O}_Y} T) = 0.$$
The conclusion that $\mathcal{N}$ is a projective object in $\mathrm{Per}^0(Y/X)$ follows by an analogous argument as the one used in Proposition 2.3.7. $\square$

Remark 2.3.9. The previous argument in Proposition 2.3.7 actually proved a stronger result. Indeed, we proved that, for $\mathcal{M} \in \mathfrak{V}$ and $E \in \mathrm{Per}^{-1}(Y/X)$,
$$\mathrm{Hom}_{\mathrm{Per}^{-1}(Y/X)}(\mathcal{M}, E) = \mathbf{H}^0(\mathbf{R}\,\mathrm{Hom}_Y(\mathcal{M}, E)).$$
and that
$$\mathbf{H}^i(\mathbf{R}\,\mathrm{Hom}_Y(\mathcal{M}, E)) = 0 \text{ for } i \neq 0.$$
Therefore, $\mathbf{R}\,\mathrm{Hom}_Y(\mathcal{M}, E) = \mathbf{H}^0(\mathbf{R}\,\mathrm{Hom}_Y(\mathcal{M}, E))$ as objects in $D^b(R)$. In conclusion, we obtained
$$\mathrm{Hom}_{\mathrm{Per}^{-1}(Y/X)}(\mathcal{M}, E) = \mathbf{R}\,\mathrm{Hom}_Y(\mathcal{M}, E)$$
in $D^b(R)$. Similarly, we get that for $\mathcal{N} \in \mathfrak{V}^\vee$ and $E \in \mathrm{Per}^0(Y/X)$,
$$\mathrm{Hom}_{\mathrm{Per}^0(Y/X)}(\mathcal{N}, E) = \mathbf{R}\,\mathrm{Hom}_Y(\mathcal{N}, E)$$
in $D^b(R)$.

Remark 2.3.10. The structure sheaf $\mathcal{O}_Y$ is by definition generated by global sections. Moreover, $\mathcal{O}_Y^\vee = \mathcal{O}_Y$, therefore the Remark 2.3.4 implies that $H^1(Y, \mathcal{O}_Y^\vee) = 0$. This proves that $\mathcal{O}_Y$ belongs both to $\mathfrak{V}$ and to $\mathfrak{V}^\vee$, hence by Propositions 2.3.7 and 2.3.8 it is both a projective object in $\mathrm{Per}^{-1}(Y/X)$ and in $\mathrm{Per}^0(Y/X)$.

We want to characterize the projective generators in $\mathrm{Per}^p(Y/X)$. First, we recall some equivalent definitions of projective generators and their main properties.

Definition 2.3.11. Let $\mathcal{C}$ be an abelian category and let $\mathcal{P} \in \mathcal{C}$ be a projective object. We say that $\mathcal{P}$ is a *projective generator* if it satisfies one of the following equivalent conditions.

(a) If $M \in \mathcal{C}$ is such that $\mathrm{Hom}(\mathcal{P}, M) = 0$, then $E = 0$.

(b) Every object $M \in \mathcal{C}$ admits an epimorphism $\mathcal{P}^{\oplus I} \twoheadrightarrow M$.

(c) If $f \in \mathrm{Hom}(M, N)$ is a nonzero morphism, then there exists $g \in \mathrm{Hom}(P, M)$ such that $f \circ g \neq 0$.

Proposition 2.3.12 ([Wil65, Theorem 2.3])**.** Let $\mathcal{C}$ be an abelian category. The following properties are equivalent for an object $\mathcal{P} \in \mathcal{C}$.

(i) $\mathcal{P}$ is a projective generator.

(ii) For all objects $M \in \mathcal{C}$, there is an equality $M = \sum_{f \in \operatorname{Hom}(\mathcal{P}, M)} f(\mathcal{P})$.

(iii) If $\mathcal{Q} \in \mathcal{C}$ is a projective object, then there exists another object $\mathcal{Q}' \in \mathcal{C}$ (which must then be projective) such that
$$\mathcal{Q} \oplus \mathcal{Q}' \simeq \bigoplus_I \mathcal{P}.$$

Remark 2.3.13. The case we are interested in is $\mathcal{C} = \operatorname{Per}^p(Y/X)$. In particular, we are working in the derived category of coherent sheaves on a Noetherian scheme. In this setting, the index set I can be taken to be finite, both in Definition 2.3.11(b) and in Proposition 2.3.12(iii).

Lemma 2.3.14. Let $f \colon Y \to X$ be a projective morphism between Noetherian schemes with fibers of dimension at most n, suppose that X is affine. Fix an ample line bundle $\mathcal{L}$ on Y and $a \in \mathbb{Z}$.

If $M \in D^b(\operatorname{coh}(Y))$ is such that $\operatorname{Ext}^i_Y(\mathcal{L}^{a+j}, M) = 0$ for all i and for $0 \leq j \leq n$, then $M = 0$.

Proof. Assume without loss of generality that $a = -n$. The ample line bundle $\mathcal{L}$ defines a map $Y \to \mathbb{P}^N_X$. Use the Koszul complex of a polynomial ring in $N+1$ variables to construct the following exact sequence of vector bundles on $\mathbb{P}^N_X$
$$0 \to \mathcal{O}_{\mathbb{P}^N_X}(-N-1) \to \cdots \to \mathcal{O}_{\mathbb{P}^N_X}(-u)^{\binom{N+1}{u}} \to \cdots \to \mathcal{O}_{\mathbb{P}^N_X} \to 0.$$
Taking the inverse image on Y, we get an exact sequence
$$0 \to \mathcal{L}^{-N-1} \to \cdots \to (\mathcal{L}^{-u})^{\oplus \binom{N+1}{u}} \to \cdots \to \mathcal{O}_Y \to 0.$$
Then, by taking the kernel at $(\mathcal{L}^{-n-1})^{\oplus \binom{N+1}{n+1}}$, we get the exact sequence
$$0 \to K \to (\mathcal{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to \cdots \to (\mathcal{L}^{-1})^{\oplus N+1} \to \mathcal{O}_Y \to 0,$$
which represents an element of $\operatorname{Ext}^{n+1}_Y(\mathcal{O}_Y, K)$. But $\operatorname{Ext}^{n+1}_Y(\mathcal{O}_Y, K) = 0$ by Lemma 2.3.3, so the extension is trivial.
On the other hand, $\operatorname{Ext}^{n+1}_Y(\mathcal{O}_Y, K) = \operatorname{Hom}_{D^b(\operatorname{coh}(Y))}(\mathcal{O}_Y, K[n+1]) = \operatorname{Ext}^1_Y(\mathcal{O}_Y, K[n])$, so the above trivial extension can be seen as a direct sum in the derived category. This proves that $\mathcal{O}_Y$ is a direct summand in $D^b(\operatorname{coh}(Y))$ of the complex
$$(\mathcal{L}^{-n-1})^{\oplus \binom{N+1}{n+1}} \to \cdots \to (\mathcal{L}^{-1})^{\oplus N+1}.$$
Dualizing and tensoring with $\mathcal{L}^{-n-p-1}$, we get that $\mathcal{L}^{-n-p-1}$ is a direct summand of
$$\mathcal{L}^\bullet_{-p} := (\mathcal{L}^{-n-p})^{\oplus N+1} \to \cdots \to (\mathcal{L}^{-p})^{\oplus \binom{N+1}{n+1}}.$$
Therefore, for an object $U_p \in D^b(\operatorname{coh}(Y))$, there is an isomorphism
$$\mathcal{L}^\bullet_{-p} \simeq \mathcal{L}^{-n-p-1} \oplus \mathcal{U}_p. \tag{2.3}$$
We use this isomorphism to prove by induction on p that the following properties hold for all $p \geq 0$:

(a) $\mathrm{Ext}^i_Y(\mathcal{L}^j, M) = 0$ for all $0 \geq j \geq -n-p$,

(b) $\mathrm{Ext}^i_Y(\mathcal{L}^\bullet_{-p}, M) = 0$ for all $p \geq 0$.

We study first the case $p = 0$. Property (a) holds by hypothesis. For property (b), we use the spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^i_Y((\mathcal{L}^\bullet_0)^j, M) \Longrightarrow \mathrm{Ext}^i_Y(\mathcal{L}^\bullet_0, M).$$

Indeed, $\mathrm{Ext}^i_Y((\mathcal{L}^\bullet_0)^j, M) = 0$ for all i, j by hypothesis, so the second property is proven.

Suppose now $p > 0$. Using the isomorphism (2.3), we get

$$\mathrm{Ext}^i_Y(\mathcal{L}^\bullet_{-p}, M) \simeq \mathrm{Ext}^i_Y(\mathcal{L}^{-n-p-1}, M) \oplus \mathrm{Ext}^i_Y(\mathcal{U}_p, M).$$

But $\mathrm{Ext}^i_Y(\mathcal{L}^\bullet_{-p}, M)$ vanishes by inductive hypothesis, thus proving $\mathrm{Ext}^i_Y(\mathcal{L}^{-n-p-1}, M) = 0$. Joined with the inductive hypothesis, this proves property (a).

For property (b), we use once again the spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^i_Y((\mathcal{L}^\bullet_{-p-1})^j, M) \Longrightarrow \mathrm{Ext}^i_Y(\mathcal{L}^\bullet_{-p-1}, M),$$

where $\mathrm{Ext}^i_Y((\mathcal{L}^\bullet_{-p-1})^j, M)$ vanishes by (a).

In conclusion, we proved $\mathrm{Ext}^i_Y(\mathcal{L}^j, M) = 0$ for all i and for all $j \leq 0$. Applying this with $i = 0$ and using the fact that $\mathcal{L}$ is ample, [Har13, §II, Corollary 5.18] implies that $M = 0$.

□

Proposition 2.3.15. There exists a vector bundle $\mathcal{P} \in \mathfrak{V}$ which is a projective generator in $\mathrm{Per}^{-1}(Y/X)$ and whose dual $\mathcal{P}^\vee$ is a projective generator in $\mathrm{Per}^0(Y/X)$.

Proof. let $\mathcal{L}$ be an ample line bundle on Y generated by global sections and let $r-1$ be the rank of $H^1(Y, \mathcal{L}^\vee)$ over R. We have the following isomorphisms

$$H^1(Y, \mathcal{L}^\vee)^{\oplus r-1} \simeq \mathrm{Ext}^1_Y(\mathcal{L}, \mathcal{O}_Y)^{\oplus r-1} \simeq \mathrm{Ext}^1_Y(\mathcal{L}, \mathcal{O}_Y^{\oplus r-1}).$$

Therefore, a set of $r-1$ generators of $H^1(Y, \mathcal{L}^\vee)$ gives rise to an extension

$$0 \to \mathcal{O}_Y^{\oplus r-1} \to \mathcal{P}_0 \to \mathcal{L} \to 0. \tag{2.4}$$

Notice that $\mathcal{P}_0$ is a vector bundle, as it is an extension of vector bundles. Observe that $\mathcal{P}_0$ belongs to $\mathfrak{V}$, as we shall show now.

- By dualizing (2.4) and then taking the long exact sequence in cohomology, we get

$$H^0(Y, \mathcal{O}_Y)^{\oplus r-1} \to H^1(Y, \mathcal{L}^\vee) \to H^1(Y, \mathcal{P}_0^\vee) \to 0.$$

  In addition, there is an R-modules isomorphism $H^0(Y, \mathcal{O}_Y)^{\oplus r-1} \simeq R^{r-1}$. The first map is then obtained by mapping the canonical generators of R^{r-1} to the fixed generators of $H^1(Y, \mathcal{L}^\vee)$. In particular it is surjective, thus implying $H^1(Y, \mathcal{P}_0^\vee) = 0$.

- By taking the long exact sequence in cohomology obtained from (2.4) and tensoring with $\mathcal{O}_Y$, we get

$$H^0(Y,\mathcal{O}_Y)^{\oplus r-1} \otimes_R \mathcal{O}_Y \to H^0(Y,\mathcal{P}_0) \otimes_R \mathcal{O}_Y \to H^0(Y,\mathcal{L}) \otimes_R \mathcal{O}_Y \to 0.$$

  Consider the following commutative diagram with exact rows

$$\begin{tikzcd} & H^0(Y,\mathcal{O}_Y)^{\oplus r-1} \otimes_R \mathcal{O}_Y \arrow[r] \arrow[d] & H^0(Y,\mathcal{P}_0) \otimes_R \mathcal{O}_Y \arrow[r] \arrow[d] & H^0(Y,\mathcal{L}) \otimes_R \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_Y^{\oplus r-1} \arrow[r] & \mathcal{P}_0 \arrow[r] & \mathcal{L} \arrow[r] & 0 \end{tikzcd}$$

  Since $\mathcal{O}_Y$ and $\mathcal{L}$ are both globally generated, the first and last columns are actually surjective. Therefore

$$H^0(Y,\mathcal{P}_0) \otimes_R \mathcal{O}_Y \to \mathcal{P}_0$$

  is surjective, i.e. $\mathcal{P}_0$ is generated by global sections. This proves that $\mathcal{P}_0 \in \mathfrak{V}$.

Put $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{O}_Y$. Then $\mathcal{P} \in \mathfrak{V}$ and hence by Lemma 2.3.7 $\mathcal{P}$ is a projective object in $\mathrm{Per}^{-1}(Y/X)$.

We are left to show that $\mathcal{P}$ is a generator. Let $E \in \mathrm{Per}^{-1}(Y/X)$ be such that $\mathrm{Ext}^i_Y(\mathcal{P},E) = 0$ for all i. Then in particular $\mathrm{Ext}^i_Y(\mathcal{O}_Y,E) = 0$ and $\mathrm{Ext}^i_Y(\mathcal{L},E) = 0$ for all i. Then the conclusion that $\mathcal{P}$ is a generator follows from Lemma 2.3.14. The proof for $\mathcal{P}^\vee$ is analogous. □

Remark 2.3.16. Since determinant is multiplicative along short exact sequences, the vector bundle $\mathcal{P}$ constructed in Proposition 2.3.15 has the property

$$\det(\mathcal{P}) = \det(\mathcal{P}_0) = \mathcal{L}.$$

In particular, $\det(\mathcal{P})$ is ample.

Proposition 2.3.17. The projective objects in $\mathrm{Per}^{-1}(Y/X)$ are exactly the objects in $\mathfrak{V}$. The projective objects in $\mathrm{Per}^0(Y/X)$ are exactly the objects in $\mathfrak{V}$.

Proof. We study the case of $\mathrm{Per}^{-1}(Y/X)$. By Proposition 2.3.7, the objects in $\mathfrak{V}$ are projective. Moreover, Proposition 2.3.15 shows that $\mathfrak{V}$ contains a projective generator. But $\mathfrak{V}$ is closed under direct sums and direct summands, so Proposition 2.3.12(iii) shows that $\mathfrak{V}$ contains every projective object. The proof for $\mathrm{Per}^0(Y/X)$ is the same. □

We are finally ready to characterize the projective generators in $\mathrm{Per}^p(Y/X)$.

If $\mathcal{M}$ is a vector bundle of rank r on Y, we denote $\bigwedge^r \mathcal{M}$ by $\det(\mathcal{M})$.

Proposition 2.3.18. The projective generators in $\mathrm{Per}^{-1}(Y/X)$ are the vector bundles $\mathcal{M} \in \mathfrak{V}$ such that $\det(\mathcal{M})$ is an ample line bundle and such that $\mathcal{O}_Y$ is a direct summand of some $\mathcal{M}^{\oplus a}$. The projective generators in $\mathrm{Per}^0(Y/X)$ are the vector bundles $\mathcal{N} = \mathcal{M}^\vee \in \mathfrak{V}^\vee$ such that $\mathcal{M}$ is a projective generator in $\mathrm{Per}^{-1}(Y/X)$.

Proof. Let $\mathcal{M}$ be a projective generator in $\mathrm{Per}^{-1}(Y/X)$. The fact that $\mathcal{O}_Y$ is a direct summand of $\mathcal{M}^{\oplus a}$ is clear. Indeed, $\mathcal{O}_Y$ is a projective object in $\mathrm{Per}^{-1}(Y/X)$, so this is just Proposition 2.3.12(iii). We then have to prove that $\det(\mathcal{M})$ is ample.

Let $\mathcal{P}$ be projective generator constructed in Proposition 2.3.15. Using again Proposition 2.3.12(iii), we find that

$$\mathcal{M}^{\oplus b} = \mathcal{P} \oplus \mathcal{P}',$$

with $b \in \mathbb{N}$. Then
$$\det(\mathcal{M})^{\otimes b} = \det(\mathcal{P}) \otimes \det(\mathcal{P}').$$
Notice that $\det(\mathcal{P})$ is ample by construction and $\det(\mathcal{P}')$ is generated by global sections since $\mathcal{P}' \in \mathfrak{V}$. Then by [Har13, §II, Exercise 7.5.] $\det(\mathcal{M})^{\otimes b}$ is ample and hence by [Har13, §II, Proposition 7.5.] $\det(\mathcal{M})$ is also ample.

Suppose now that $\mathcal{M} \in \mathfrak{V}$ is as stated. Since $\mathcal{M} \in \mathfrak{V}$, we know that it is projective. We prove that it is a generator. Let $E \in \mathrm{Per}^{-1}(Y/X)$ be such that $\mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathcal{M}, E) = \mathrm{Hom}_Y(\mathcal{M}, E) = 0$. Then
$$\mathrm{Ext}^i_Y(\mathcal{M}, E) = 0 \text{ for all } i \geq 0,$$
since the case $i > 0$ follows from the fact that $\mathcal{M} \in \mathfrak{V}$ and $E \in \mathrm{Per}^{-1}(Y/X)$.

Using the spectral sequence
$$E_2^{p,q} := \mathrm{Ext}^p_Y(\mathcal{M}, \mathbf{H}^q(E)) \Longrightarrow \mathrm{Ext}^{p+q}_Y(\mathcal{M}, E)$$
we get that
$$\mathrm{Ext}^i_Y(\mathcal{M}, \mathbf{H}^j(E)) = 0 \text{ for all } i, j.$$
On the other hand, if $\mathbf{H}^j(E) = 0$ for all j, then $E = 0$ in $\mathrm{Per}^{-1}(Y/X)$. In conclusion, it suffices to show that if $E \in \mathrm{coh}(Y)$ is a coherent sheaf such that $\mathrm{Ext}^i_Y(\mathcal{M}, E) = 0$ for all i, then $E = 0$.

Suppose then that $E \in \mathrm{coh}(Y)$. Considering E as an R-module and using Nakayama's Lemma, it suffices to prove that $E/\mathfrak{M}E = 0$ for all maximal ideals $\mathfrak{M} \subseteq R$. Indeed,
$$E/\mathfrak{M}E \simeq E/(f^*(\mathcal{I}_x) \otimes_{\mathcal{O}_Y} E) \simeq E \otimes_{\mathcal{O}_Y} f^*\mathcal{O}_x$$
for any closed point $x \in X$ associated to $\mathfrak{M}$. If the restriction of E to each closed fiber of f is zero, then E must be zero.

First notice that, since $\mathfrak{M}E \subseteq E$, we have $\mathrm{Hom}_Y(\mathcal{M}, \mathfrak{M}E) = 0$. Moreover, by Noetherianity of R, we can find a short exact sequence
$$0 \to K \to E^{\oplus c} \to \mathfrak{M}E \to 0,$$
which yields the long exact sequence
$$0 = \mathrm{Ext}^1_Y(\mathcal{M}, E^{\oplus c}) \to \mathrm{Ext}^1_Y(\mathcal{M}, \mathfrak{M}E) \to \mathrm{Ext}^2(\mathcal{M}, K) = 0$$
This proves that $\mathrm{Ext}^1_Y(\mathcal{M}, \mathfrak{M}E) = 0$. Then, by Lemma 2.3.3, we conclude that
$$\mathrm{Ext}^i_Y(\mathcal{M}, \mathfrak{M}E) = 0 \quad \text{for all } i \geq 0.$$
By looking at the long exact Ext sequence obtained from
$$0 \to \mathfrak{M}E \to E \to E/\mathfrak{M}E \to 0$$
we can conclude that
$$\mathrm{Ext}^i_Y(\mathcal{M}, E/\mathfrak{M}E) = 0 \quad \text{for all } i \geq 0.$$
Let $x \in X$ be the point whose defining ideal is $\mathfrak{M}$ and let $C \subseteq Y$ be fiber over x. Then we have the cartesian square

$$
\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "f|_C"] & Y \arrow[d, "f"] \\
x \arrow[r, "j"] & X.
\end{tikzcd}
$$

Notice that $E/\mathfrak{M}E = i_*(E|_C) = i_*i^*E$. Then

$$\operatorname{Ext}^i_Y(\mathcal{M}, E/\mathfrak{M}E) = \operatorname{Ext}^i_Y(\mathcal{M}, i_*i^*E) = \operatorname{Ext}^i_C(i^*\mathcal{M}, i^*E) = \operatorname{Ext}^i_C(\mathcal{M}/\mathfrak{M}\mathcal{M}, E/\mathfrak{M}E).$$

Therefore, it suffices to show that $E|_C = 0$ if $\operatorname{Ext}^i_C(\mathcal{M}|_C, E|_C) = 0$ for all i. In conclusion, we are left to prove that if X is the spectrum of a field and E is a coherent sheaf on Y such that $\operatorname{Ext}^i_Y(\mathcal{M}, E) = 0$ for all i, then $E = 0$.

By Lemma 2.5.12, we have the short exact sequence

$$0 \to \mathcal{O}_Y^{r-1} \to \mathcal{M} \to \mathcal{L} \to 0,$$

where $\mathcal{L} = \det(\mathcal{M})$ is an ample line bundle generated by global sections. Since $\mathcal{O}_Y$ is a direct summand of $\mathcal{M}^{\oplus a}$, the hypothesis $\operatorname{Ext}^i_Y(\mathcal{M}, E) = 0$ implies $\operatorname{Ext}^i_Y(\mathcal{O}_Y, E) = 0$ for all i and therefore $\operatorname{Ext}^i_Y(\mathcal{L}, E) = 0$ for all i. Then we conclude $E = 0$ by applying Lemma 2.3.14.

The case of $\operatorname{Per}^0(Y/X)$ is similar. □

We now come to the main result of this section. We establish the existence of a derived equivalence

$$D^b(\operatorname{coh}(Y)) \xrightarrow{\simeq} D^b(A), \tag{2.5}$$

where A is a k-algebra. The equivalence (2.5) is constructed to be t-exact, if $D^b(\operatorname{coh}(Y))$ is endowed with the perverse t-structure and $D^b(A)$ with the standard t-structure. Therefore, it restricts to an equivalence

$$\operatorname{Per}^p(Y/X) \xrightarrow{\simeq} \operatorname{mod}(A)$$

between the hearts of the respective t-structures.

In particular, we need to construct two t-exact functors between the derived categories and to prove that they are equivalences. For the latter, we use the following general statement, which can be found in [Bas68, §II, Theorem 1.3].

Proposition 2.3.19. *Let R be a Noetherian commutative ring and let $\mathcal{C}$ be an R-linear category such that $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is a finitely generated R-module for all $A, B \in \mathcal{C}$.*

Let $\mathcal{P} \in \mathcal{C}$ be a projective generator and let A be the R-algebra $A := \operatorname{End}_{\mathcal{C}}(\mathcal{P})$. Then the functors

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{P}, -) \colon \mathcal{C} \longrightarrow \operatorname{mod}(A),$$
$$- \otimes_A \mathcal{P} \colon \operatorname{mod}(A) \longrightarrow \mathcal{C}$$

define inverse equivalences, where $\operatorname{mod}(A)$ is the category of right finitely generated A-modules.

Theorem 2.3.20. *Assume that $\mathcal{P}$ is a projective generator in $\operatorname{Per}^p(Y/X)$ and let $A := \operatorname{End}_Y(\mathcal{P})$. In order to distinguish the R-module structure and the A-module structure on $\mathcal{P}$, we denote the latter by ${}_A\mathcal{P}$. Then the functors*

$$\mathbf{R}\operatorname{Hom}_Y({}_A\mathcal{P}, -) \colon D^b(\operatorname{coh}(Y)) \longrightarrow D^b(A),$$
$$- \overset{\mathbf{L}}{\otimes}_A {}_A\mathcal{P} \colon D^b(A) \to D^b(\operatorname{coh}(Y))$$

define inverse equivalences of triangulated categories. These equivalences map the perverse t-structure of $D^b(\mathrm{coh}(Y))$ to the canonical t-structure of $D^b(A)$. Therefore, they restrict to equivalences between $\mathrm{Per}^p(Y/X)$ and $\mathrm{mod}(A)$.

Proof. We prove the second statement and derive the first by induction over triangles, as in Lemma 2.3.21 below. Thus, we need to prove both that the functors map $\mathrm{Per}^p(Y/X)$ to $\mathrm{mod}(A)$ and vice versa and that the restrictions to these subcategories are actually the functors appearing in Proposition 2.3.19, hence they are equivalences.

Notice that the functors $\mathbf{R}\,\mathrm{Hom}_Y({}_A\mathcal{P},-)$ and $\mathbf{R}\,\mathrm{Hom}_Y({}_R\mathcal{P},-)$ are both computed by choosing an injective resolution in $D^b(\mathrm{coh}(Y))$. Therefore, $\mathbf{R}\,\mathrm{Hom}_Y({}_R\mathcal{P},-)$ can be obtained by composing $\mathbf{R}\,\mathrm{Hom}_Y({}_A\mathcal{P},-)$ with the forgetful functor $D^b(A) \to D^b(R)$. In conclusion, we can denote both of them simply by $\mathbf{R}\,\mathrm{Hom}_Y(\mathcal{P},-)$.

It was already observed in Remark 2.3.9 that

$$\mathrm{Hom}_{\mathrm{Per}^p(Y/X)}(\mathcal{P},-) = \mathbf{R}\,\mathrm{Hom}_Y(\mathcal{P},-)|_{\mathrm{Per}^p(Y/X)}.$$

Actually, we observed this as an equality in $D^b(R)$, but as noted above it is also an equality in $D^b(A)$. Therefore, the statement regarding the first functor is proven.

Next we study the restriction of the functor $- \overset{\mathbf{L}}{\otimes}_A {}_A\mathcal{P}$ to $\mathrm{mod}(A)$. Let M be a finitely generated A-module. To compute the derived tensor product, we can use a free resolution $F^\bullet$ of M consisting of finite rank modules. Then

$$M \overset{\mathbf{L}}{\otimes}_A {}_A\mathcal{P} = F^\bullet \otimes_A {}_A\mathcal{P}. \tag{2.6}$$

On the other hand, considering $\mathcal{P}$ as an object in the perverse category $\mathrm{Per}^p(Y/X)$, we can study the functors defined in Proposition 2.3.19. In order to keep in mind this distinction, denote the tensor product functor on $\mathrm{Per}^p(Y/X)$ as

$$-^p \otimes_A {}_A\mathcal{P} \colon \mathrm{mod}(A) \longrightarrow \mathrm{Per}^p(Y/X).$$

Following this notation, we can compute cohomology in the perverse category and in particular use the same free resolution to compute perverse Tor as

$${}^p\mathrm{Tor}_i^A(M,\mathcal{P}) := {}^p\mathbf{H}^{-i}(F^{\bullet p} \otimes_A {}_A\mathcal{P}).$$

By Proposition 2.3.19, the functor $-^p \otimes_A {}_A\mathcal{P}$ is exact, thus implying that ${}^p\mathrm{Tor}_i^A(M,\mathcal{P}) = 0$ for $i > 0$. But since $F^\bullet$ is a complex of free modules, it is clear that

$$F^\bullet \otimes_A {}_A\mathcal{P} = F^{\bullet p} \otimes_A {}_A\mathcal{P}. \tag{2.7}$$

Joining the relations (2.6) and (2.7), we deduce that

$$M \overset{\mathbf{L}}{\otimes}_A {}_A\mathcal{P} = F^{\bullet p} \otimes_A {}_A\mathcal{P}.$$

But since $-^p \otimes_A \mathcal{P}$ is exact and does not need to be derived, actually

$$M \overset{\mathbf{L}}{\otimes}_A {}_A\mathcal{P} = M^p \otimes_A {}_A\mathcal{P}.$$

This proves that $- \overset{\mathbf{L}}{\otimes}_A {}_A\mathcal{P}$ restricted to $\mathrm{mod}(A)$ coincides with $-^p \otimes_A {}_A\mathcal{P}$. Therefore it defines an equivalence between $\mathrm{mod}(A)$ and $\mathrm{Per}^p(Y/X)$, as required. $\square$

Lemma 2.3.21. Let $\mathcal{C}, \mathcal{D}$ be triangulated categories endowed with t-structures $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$, $(\mathcal{C}^{\geq 0}, \mathcal{C}^{\leq 0})$. Denote the truncation functors by $\tau^{\mathcal{C}}, \tau^{\mathcal{D}}$ and the hearts by $\mathcal{H}^{\mathcal{C}}, \mathcal{H}^{\mathcal{D}}$.
Suppose that for all objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ there exist $a, b \in \mathbb{N}$ (depending on the chosen objects) such that $X \in \mathcal{D}^{[a,b]}$ and $Y \in \mathcal{C}^{[a,b]}$.
If $F\colon \mathcal{D} \to \mathcal{C}$ is an exact functor mapping $\mathcal{H}^{\mathcal{D}}$ to $\mathcal{H}^{\mathcal{C}}$ such that $F|_{\mathcal{H}^{\mathcal{D}}}\colon \mathcal{H}^{\mathcal{D}} \to \mathcal{H}^{\mathcal{C}}$ is an equivalence, then F is an equivalence.

Proof. We prove by induction on $n = b - a$ that $F|_{\mathcal{D}^{[a,b]}}\colon \mathcal{D}^{[a,b]} \to \mathcal{C}^{[a,b]}$ is an equivalence. Since $\mathcal{D}^{[0,0]} = \mathcal{H}^{\mathcal{D}}$ and $\mathcal{C}^{[0,0]} = \mathcal{H}^{\mathcal{C}}$, the first step of the induction is trivial.

For the inductive step, let $X \in \mathcal{D}^{[a,b]}$ and consider the triangle

$$\tau^{\mathcal{D}}_{\leq b-1} X \to X \to \tau^{\mathcal{D}}_{\geq b} X.$$

Applying $\tau^{\mathcal{D}}_{\leq b}$ we get

$$\tau^{\mathcal{D}}_{\leq b} \tau^{\mathcal{D}}_{\leq b-1} X \to \tau^{\mathcal{D}}_{\leq b} X \to \tau^{\mathcal{D}}_{\leq b} \tau^{\mathcal{D}}_{\geq b} X.$$

Observe that

- $\tau^{\mathcal{D}}_{\leq b} \tau^{\mathcal{D}}_{\leq b-1} X = \tau^{\mathcal{D}}_{\leq b-1} X$ belongs to $\mathcal{D}^{[a,b-1]}$. Therefore, the inductive hypothesis applies.
- $\tau^{\mathcal{D}}_{\leq b} \tau^{\mathcal{D}}_{\geq b} X$ belongs to $\mathcal{D}^{[b,b]} = \mathcal{H}^{\mathcal{D}}[b]$.

Since $\tau^{\mathcal{D}}_{\leq b} X = X$, this proves that $F|_{\mathcal{D}^{[a,b]}}$ is an equivalence. $\square$

We mention a final result that holds in the affine case, which will apply to the case studied in Section 2.5 and which will prove useful in Section 2.6.

Lemma 2.3.22 ([Van04, Lemma 3.2.9])**.** Suppose that the ring R is finitely generated over a field, or that it is a complete local ring containing a copy of its residue field. Suppose moreover that X and Y are Gorenstein of pure dimension n. If $\mathcal{M}$ is a vector bundle on Y, then for any maximal ideal $\mathfrak{M}$ in R we have $\operatorname{depth}_{\mathfrak{M}} \Gamma(Y, \mathcal{M}) \geq n - 1$. If in addition $H^1(Y, \mathcal{M}) = H^1(Y, \mathcal{M}^\vee) = 0$, then $\Gamma(Y, \mathcal{M})$ is a Cohen-Macaulay R-module.

2.4 General base

We come to the main result of the chapter, that is a global version of the derived equivalence proved in Theorem 2.3.20.

We go back to the general hypothesis specified at the beginning of the chapter, so X will be a Noetherian equidimensional scheme. In this setting, the role of the projective generators is replaced by a local analogous.

Definition 2.4.1. An object $\mathcal{P} \in \operatorname{Per}^p(Y/X)$ is called a *local projective generator* if X admits an open covering $X = \cup_i X_i$ such that

- X_i is affine open;
- $\mathcal{P}|_{f^{-1}(X_i)}$ is a projective generator in $\operatorname{Per}^p(f^{-1}(X_i)/X_i)$ for all i.

The following theorem is an immediate consequence of Theorem 2.3.20, proven by restriction to the affine opens X_i.

Theorem 2.4.2. Assume that $\mathcal{P}$ is a local projective generator in $\mathrm{Per}^p(Y/X)$ and let $\mathcal{A} := f_*\mathcal{E}nd_Y(\mathcal{P})$. Then the functors

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_Y(\mathcal{P}, -)\colon D^b(\mathrm{coh}(Y)) \longrightarrow D^b(\mathrm{coh}(\mathcal{A})),$$
$$f^{-1}(-) \overset{\mathbf{L}}{\otimes}_{f^{-1}(\mathcal{A})} \mathcal{P}\colon D^b(\mathrm{coh}(\mathcal{A})) \to D^b(\mathrm{coh}(Y))$$

define inverse equivalences of triangulated categories. These equivalences restrict to equivalences between $\mathrm{Per}^p(Y/X)$ and $\mathrm{coh}(\mathcal{A})$.

The last result needed to apply Theorem 2.4.2 is the existence of a local projective generator. This can be proven under slightly stronger hypothesis.

Proposition 2.4.3. Assume that X is quasi-projective over a Noetherian ring S. Then there exists a local projective generator $\mathcal{P}$ for $\mathrm{Per}^{-1}(Y/X)$, such that the dual $\mathcal{P}^\vee$ is a local projective generator for $\mathrm{Per}^0(Y/X)$.

Proof. Let $\bar{X}$ be a projective S-scheme containing X as an open subset. The morphism f factors as $Y \hookrightarrow \mathbb{P}^N_X \to X$. Define $\bar{Y}$ as the closure of Y under the embedding $Y \hookrightarrow \mathbb{P}^N_X \hookrightarrow \mathbb{P}^N_{\bar{X}}$ and the morphism $\bar{f}\colon \bar{Y} \to \bar{X}$ as $\bar{Y} \hookrightarrow \mathbb{P}^N_{\bar{X}} \to \bar{X}$. We first construct a local projective generator in $\mathrm{Per}^{-1}(\bar{Y}/\bar{X})$.

Let $\bar{\mathcal{L}}$ be an $\bar{f}$-ample line bundle on $\bar{Y}$ generated by global sections. Let $a, b \in \mathbb{N}$ be big enough such that, setting $\bar{\mathcal{E}} = \mathcal{O}_{\bar{X}}(-a)^{\oplus b}$, we have that

- there is a surjective morphism $\varphi\colon \bar{\mathcal{E}} \to \mathbf{R}^1\bar{f}_*(\bar{\mathcal{L}}^{-1})$,
- $\mathrm{Ext}^i_{\bar{X}}(\bar{\mathcal{E}}, \bar{f}_*\bar{\mathcal{L}}^{-1}) = 0$ for all $i > 0$.

Observe that

$$\mathrm{Ext}^1_{\bar{Y}}(\bar{\mathcal{L}}, \bar{f}^*(\bar{\mathcal{E}}^\vee)) = \mathrm{Ext}^1_{\bar{Y}}(\bar{f}^*\bar{\mathcal{E}}, \bar{\mathcal{L}}^{-1}) = \mathrm{Ext}^1_{\bar{X}}(\bar{\mathcal{E}}, \mathbf{R}\bar{f}_*\bar{\mathcal{L}}^{-1}).$$

Consider the Leray spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\bar{X}}(\bar{\mathcal{E}}, \mathbf{R}^q\bar{f}_*\bar{\mathcal{L}}^{-1}) \Longrightarrow \mathbf{H}^{p+q}(\mathbf{R}\,\mathrm{Hom}_{\bar{X}}(\bar{\mathcal{E}}, \mathbf{R}\bar{f}_*\bar{\mathcal{L}}^{-1})).$$

The assumption $\mathrm{Ext}^i_{\bar{X}}(\bar{\mathcal{E}}, \bar{f}_*\bar{\mathcal{L}}^{-1}) = 0$ for $i > 0$ yields

$$\mathrm{Ext}^1_{\bar{X}}(\bar{\mathcal{E}}, \mathbf{R}\bar{f}_*\bar{\mathcal{L}}^{-1}) = \mathbf{H}^1(\mathbf{R}\,\mathrm{Hom}_{\bar{X}}(\bar{\mathcal{E}}, \mathbf{R}\bar{f}_*\bar{\mathcal{L}}^{-1})) = \mathrm{Hom}_{\bar{X}}(\bar{\mathcal{E}}, \mathbf{R}^1\bar{f}_*\bar{\mathcal{L}}^{-1}).$$

Therefore, the morphism φ gives an extension of sheaves on $\bar{Y}$

$$0 \to \bar{f}^*(\bar{\mathcal{E}})^\vee \to \bar{\mathcal{P}}_0 \to \bar{\mathcal{L}} \to 0.$$

Restricting to $Y \subseteq \bar{Y}$, we get an extension of sheaves

$$0 \to f^*(\mathcal{E})^\vee \to \mathcal{P}_0 \to \mathcal{L} \to 0.$$

On the inverse images in Y of the affine opens in X, this extension coincides with the extension constructed in Proposition 2.3.15. Therefore $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{O}_Y$ is a local projective generator. $\square$

2.5 The formal case

In this section, we will study a particular case, that is of interest because we are able to find a more explicit expression for the projective generators.

Suppose that the base scheme X is an affine scheme $X = \operatorname{Spec} R$, such that R a Noetherian complete local ring with maximal ideal $\mathfrak{M}$. Suppose moreover that the residue field $k = R/\mathfrak{M}$ is an algebraically closed field and that $k \subseteq R$.

Let $x \in X$ be the unique closed point (corresponding to the maximal ideal $\mathfrak{M}$) and let $C = f^{-1}(x) \subseteq Y$ be the fiber over x. Under our general assumptions, C is either a single point, if f is an isomorphism, or it is a curve contained in Y. Let C_{red} be the reduced scheme structure on C.

Remark 2.5.1. C_{red} is a reduced connected projective curve over k. Therefore, $H^0(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = k$.

Lemma 2.5.2. The first cohomology group of C_{red} vanishes, i.e., $H^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = 0$.

Proof. Let $\mathcal{I}_{C_{\text{red}}} \subseteq \mathcal{O}_Y$ be the ideal sheaf of C_{red}. Since Y is Noetherian, this is a coherent sheaf and therefore, by Lemma 2.1.2, $\mathbf{R}^q f_* \mathcal{I}_{C_{\text{red}}} = 0$ for all $q \geq 2$ and $\mathbf{R}^1 f_* \mathcal{I}_{C_{\text{red}}}$ is supported at x. Moreover, $\mathbf{R}^0 f_* \mathcal{I}_{C_{\text{red}}} = f_* \mathcal{I}_{C_{\text{red}}}$ is also supported at x by the definition of C. Then, the Leray spectral sequence

$$E_2^{p,q} = H^p(X, \mathbf{R}^q f_* \mathcal{I}_{C_{\text{red}}}) \Longrightarrow H^{p+q}(X, \mathcal{I}_{C_{\text{red}}})$$

yields $H^2(Y, \mathcal{I}_{C_{\text{red}}}) = 0$. Since $H^1(Y, \mathcal{O}_Y)$ vanishes by Lemma 2.3.4, the long exact cohomology sequence obtained from

$$0 \to \mathcal{I}_{C_{\text{red}}} \to \mathcal{O}_Y \to i_* \mathcal{O}_{C_{\text{red}}} \to 0$$

gives $H^1(C_{\text{red}}, \mathcal{O}_{C_{\text{red}}}) = H^1(Y, i_* \mathcal{O}_{C_{\text{red}}}) = 0$. $\square$

Lemma 2.5.3. A reduced connected projective curve X over k of arithmetic genus 0 is a tree of $\mathbb{P}^1$'s with normal crossing.

Proof. Let $X = X_1 \cup \cdots \cup X_n$ be the decomposition of X in irreducible components. By [Liu02, §7.5, ex. 5.2], $p_a(X_i) \leq p_a(X) = 0$. On the other hand, X_i is an irreducible projective curve over an algebraically closed field, so it has nonnegative arithmetic genus, proving that $p_a(X_i) = 0$ for all i. Then the formula in [Har13, §IV, ex 1.8] proves that the normalization $\widetilde{X}_i$ has arithmetic genus zero and that X_i has no singular points. This proves in particular that $X_i = \widetilde{X}_i$ is a $\mathbb{P}^1$. Let now $\widetilde{X}$ be the normalization of X, i.e. $\widetilde{X} = \widetilde{X_1} \coprod \cdots \coprod \widetilde{X_n}$. We use [Liu02, §7.5, Proposition 5.4] and the fact that k is algebraically closed to get

$$p_a(X) + n - 1 = \sum_{i=1}^{n} p_a(\widetilde{X}_i) + \sum_{p \in \operatorname{Sing}(X)} \delta_p.$$

Under our assumptions, this becomes

$$\sum_{p \in \operatorname{Sing}(X)} \delta_p = n - 1.$$

This proves that the X_i's form a tree with normal crossings, or equivalently that X is a tree of $\mathbb{P}^1$'s. $\square$

Therefore, the following theorem is proven.

Theorem 2.5.4. The curve C_{red} is a tree of $\mathbb{P}^1$'s with normal crossings.

Lemma 2.5.5. The curve C is Cohen-Macaulay (i.e., C has no embedded components) and $H^0(C, \mathcal{O}_C) = k$.

Proof. The ideal of C in Y is $\mathfrak{M}\mathcal{O}_Y$, so we have the short exact sequence

$$0 \to \mathfrak{M}\mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_C \to 0. \tag{2.8}$$

Since $\mathfrak{M}\mathcal{O}_Y$ is generated by global sections, the vanishing of $H^1(Y, \mathcal{O}_Y)$ implies the vanishing of $H^1(Y, \mathfrak{M}\mathcal{O}_Y)$. Then, the long exact sequence obtained from Formula (2.8) shows that $H^0(C, \mathcal{O}_C)$ is a quotient of R by a proper ideal containing $\mathfrak{M}$, so it must be k. Any embedded component of C would be zero-dimensional and hence would give rise to extra sections. So such embedded components cannot exist. $\square$

Let $(C_i)_{i=1,\dots,n}$ be the irreducible components of C. By Theorem 2.5.4, each of the C_i's is a rational curve, i.e. a curve whose reduction is $\mathbb{P}^1$.

We want to use the C_i's to compute the Picard group of Y. In order to do this, it will prove useful to reduce to the formal completion of Y along the curve C. To study this case, we will need the following algebraic result.

Lemma 2.5.6. Let A be a Noetherian ring and $I \subseteq A$ an ideal. Let $\widehat{A}$ be the I-adic completion of A. Denote by π natural projection $\pi\colon \widehat{A} \to A/I$, which restricts to a group homomorphism on the units $\pi'\colon (\widehat{A})^* \to (A/I)^*$. Then $\ker \pi' = 1 + I\widehat{A}$.

Proof. An element $a \in \widehat{A}$ can be represented as

$$a = \sum_{i \geq 0} a_i, \quad a_0 \in A,\ a_i \in I^i \text{ for all } i \geq 1.$$

Notice that such a representation is unique if we require that $a_i \in I^i \smallsetminus I^{i+1}$. The elements of $\widehat{I} = I\widehat{A}$ are the ones with $a_0 = 0$. Under this representation, $\pi(a)$ is the class of a_0 in A/I.

Let a be an element of $\ker(\pi')$, i.e. $\pi(a) = 1$. This means that $a_0 = 1 + x$ for some $x \in I$. Then

$$a = 1 + x + \sum_{i \geq 1} a_i \in 1 + \widehat{I}.$$

This proves that $\ker(\pi') \subseteq 1 + \widehat{I}$.

On the other hand, using the above representation, it is clear that if $a \in 1 + \widehat{I}$, then $\pi(a) = 1$. In order to conclude the thesis, it suffices to show that $1 + \widehat{I} \subseteq (\widehat{A})^*$. Therefore, consider

$$a = 1 + \sum_{i \geq 1} a_i.$$

We define recursively $b = \sum_{i \geq 0} b_i \in \widehat{A}$ an inverse of a, as follows. For $i = 0$, let $b_0 = 1$. For $i > 0$, let

$$b_i = -\sum_{j=1}^{i} a_j b_{i-j}.$$

Notice that it follows from the recursion that $b_i \in I^i$, so b is a well defined element of $\widehat{A}$. Moreover, as a consequence of the definition we have

$$\sum_{j=0}^{i} a_j b_{i-j} = b_i + \sum_{j=1}^{i} a_j b_{i-j} = 0 \quad \text{for all } i > 0.$$

Therefore

$$a \cdot b = \left(\sum_{i\geq 0} a_i\right) \cdot \left(\sum_{i\geq 0} b_i\right) = \sum_{i\geq 0} \left(\sum_{j=0}^{i} a_j b_{i-j}\right) = a_0 b_0 = 1.$$

So b is an inverse of a in $\widehat{A}$ and the thesis is proven. $\square$

Remark 2.5.7. The previous lemma can be restated as the existence of a short exact sequence of groups

$$0 \to I\widehat{A} \to (\widehat{A})^* \to (A/I)^* \to 0. \tag{2.9}$$

This is indeed the statement used in the proof of the following theorem.

Theorem 2.5.8. The map $\mathcal{L} \mapsto (\deg(\mathcal{L} \mid C_1), \dots, \deg(\mathcal{L} \mid C_n))$ defines an isomorphism $\mathrm{Pic}(Y) \simeq \mathbb{Z}^n$.

Proof. Let $\widehat{Y}$ be the formal completion of Y along C. In particular, $\mathcal{O}_{\widehat{Y}}$ is obtained as the completion of $\mathcal{O}_Y$ along the sheaf of ideals $\mathcal{I} = \mathfrak{M}\mathcal{O}_Y$. As observed in Formula (2.9), we get a short exact sequence of sheaves

$$0 \to \mathfrak{M}\mathcal{O}_{\widehat{Y}} \to \mathcal{O}^*_{\widehat{Y}} \to \mathcal{O}^*_C \to 0.$$

As in the proof of Lemma 2.5.5, we get that $H^1(\widehat{Y}, \mathfrak{M}\mathcal{O}_{\widehat{Y}}) = 0$. This implies that $\mathrm{Pic}(\widehat{Y}) = \mathrm{Pic}(C)$. By Theorem 2.5.4, we get that $\mathrm{Pic}(C) = \mathbb{Z}^n$. On the other hand, Groethendieck's existence theorem [Stacks, Tag 089N, Theorem 76.42.11] yields $\mathrm{Pic}(Y) = \mathrm{Pic}(\widehat{Y})$. So in conclusion $\mathrm{Pic}(Y) = \mathbb{Z}^n$. The fact that the isomorphism has the indicated form follows easily by explicitly expressing all the indicated isomorphisms. $\square$

This result gives a natural choice of line bundles on Y. Indeed, we can find line bundles $(\mathcal{L}_i)_{i=1,\dots n}$ such that $\deg(\mathcal{L}_i \mid C_j) = \delta_{i,j}$. More explicitly, these line bundles can be realized via a choice of divisors. Fix for each i a point $y_i \in C_i$ not lying on any other C_j or on any of the embedded components of Y (notice that by Lemma 2.5.5 the embedded components are finite and zero-dimensional).

Theorem 2.5.9. For all $i = 1, \dots n$ there exists a divisor D_i on Y such that

$$D_i \cap C_j = \begin{cases} \{y_i\} & \text{if } i = j, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. First we observe that, for a closed subscheme $D \subseteq Y$, the connected components of D are in one to one correspondence with the connected components of $D \cap C$, or equivalently that if D is connected then so is $D \cap C$.

Indeed, since f is projective, we can use Grothendieck's existence theorem, see for example [Stacks, Tag 0885], to get an exact equivalence of categories

$$\begin{aligned} \mathrm{coh}(D) &\longrightarrow \mathrm{coh}(D, \mathfrak{M}) \\ \mathcal{F} &\longmapsto \widehat{\mathcal{F}} = \varprojlim \mathcal{F}/\mathfrak{M}^n. \end{aligned}$$

Consider in particular the case $\mathcal{F} = \mathcal{O}_D$. By definition, $\widehat{\mathcal{O}_D}$ is supported in $D \cap C$. So if $D \cap C$ is disconnected, then $\widehat{\mathcal{O}_D}$ decomposes non trivially as a direct sum. But then $\mathcal{O}_D$ must decompose non trivially as a direct sum as well. This proves that if $D \cap C$ is disconnected, then so is D.

The previous observation allows us to define D_i locally. Let U_i be an affine neighborhood of y_i. Then we can choose a nonzero divisor $z \in \Gamma(U_i, \mathcal{O}_{U_i})$ such that $V(z) \cap C_i = \{y_i\}$. Let D' be closure of $V(z)$ in Y and D_i be the component of D' containing y_i. The previous discussion shows that $D_i \cap C = \{y_i\}$, so D_i has the expected property. □

Let $\operatorname{Pic}^+(Y)$ and $\operatorname{Pic}^{++}(Y)$ be the subgroups of $\operatorname{Pic}(Y)$, consisting of isomorphism classes of line bundles that are respectively globally generated and ample.

Lemma 2.5.10. The following equalities hold.

$$\operatorname{Pic}^+(Y) = \{\mathcal{L} \in \operatorname{Pic}(Y) \mid \deg(\mathcal{L} \mid C_i) \geq 0 \text{ for all } i\},$$
$$\operatorname{Pic}^{++}(Y) = \{\mathcal{L} \in \operatorname{Pic}(Y) \mid \deg(\mathcal{L} \mid C_i) > 0 \text{ for all } i\}.$$

Proof. Let $\mathcal{L}$ be globally generated. Since $\mathcal{L}$ is a line bundle, this is equivalent to $\mathcal{L}$ being basepoint-free. But then the restrictions of $\mathcal{L}$ to the curves C_i are basepoint-free as well, proving by [Har13, §IV.1, Lemma 1.2] that the degree of $\mathcal{L}$ is non negative along each of the C_i's.

Let $\mathcal{L}$ be a line bundle on Y that has nonnegative degree along each of the curves C_i. By the previous constructions, $\mathcal{L}$ can be realized as $\mathcal{O}_Y(E)$, where E is a divisor $E = \sum_{i=1}^n m_i D_i$, with $m_i \geq 0$ for all i. In particular, E is a closed subscheme of Y thus yielding the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(E) \to \mathcal{O}_E(E) \to 0. \tag{2.10}$$

Notice that E is finite over X. Indeed, $f|_E \colon E \to X$ is the composition of a closed embedding and a projective morphism, so it is proper. Moreover, it has finite fibers, since each D_i intersects C in a single point. Hence $f|_E$ is finite, proving that E is affine. This implies that the support of $\mathcal{O}_E(E)$ is affine, so $\mathcal{O}_E(E)$ is generated by global sections. Then, using the vanishing of $H^1(Y, \mathcal{O}_Y)$, the long exact sequence obtained from (2.10) proves that $\mathcal{O}_Y(E)$ is globally generated, with an analogous argument as the one present in the proof of Proposition 2.3.15.

Let $\mathcal{L}$ be an ample line bundle on Y. Then, since the closed immersion $j \colon C_i \hookrightarrow Y$ is affine, we have that $\mathcal{L}|C_i = j^*(\mathcal{L})$ is ample on C_i. Therefore, $\deg(\mathcal{L} \mid C_i) > 0$ for all i.

Finally, suppose that $\mathcal{L}$ is a line bundle on Y with positive degree along all the C_i's. Applying [Kee03, Proposition 2.7] to the morphism f, in order to prove that $\mathcal{L}$ is ample it suffices to show that $\mathcal{L}|_{f^{-1}(p)}$ is ample on $f^{-1}(p)$ for all $p \in X$. For $p \neq x$ this is trivial, since $f^{-1}(p)$ is a single point. For $p = x$, we have to show that $\mathcal{L}|C$ is ample. But the hypothesis is that $\mathcal{L}$ has positive degree on the irreducible components of C, so the conclusion follows from [Liu02, §7.5, Proposition 5.5]. □

We now aim to classify the indecomposable projective objects in $\operatorname{Per}^{-1}(Y/X)$ and $\operatorname{Per}^0(Y/X)$.

Lemma 2.5.11. Let $\mathcal{M}$ be a vector bundle of rank r on Y generated by global sections. Then $\mathcal{M}$ occurs in short exact sequences

$$0 \to \mathcal{O}_Y^{\oplus r-1} \to \mathcal{M} \to \mathcal{L} \to 0,$$
$$0 \to \mathcal{L}^{-1} \to \mathcal{O}_Y^{\oplus r+1} \to \mathcal{M} \to 0$$

where $\mathcal{L}$ is the determinant bundle $\mathcal{L} = \det(\mathcal{M})$.

Proof. Let $\mathcal{J}_i$ be the ideal sheaf in Y of the curve C_i. Notice that $H^0(Y, \mathcal{J}_i) \subsetneq R$, since $\mathcal{J}_i$ is a proper subsheaf of $\mathcal{O}_Y$. So the long exact cohomology sequence obtained from

$$0 \to \mathcal{J}_i \to \mathcal{O}_Y \to \mathcal{O}_{C_i} \to 0 \tag{2.11}$$

yields $H^1(Y, \mathcal{J}_i) = 0$.

We claim that $H^1(Y, \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{J}_i) = 0$. Since $\mathcal{M}$ is globally generated, there is a surjective morphism $H^0(Y, \mathcal{M}) \otimes_R \mathcal{O}_Y \to \mathcal{M}$. Tensoring with $\mathcal{J}_i$ we get a short exact sequence

$$0 \to \mathcal{K} \to H^0(Y, \mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{J}_i \to \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{J}_i \to 0.$$

The vanishing of $H^1(Y, \mathcal{J}_i)$ implies the vanishing of the first cohomology group of the central term. Moreover, $H^2(Y, \mathcal{K})$ vanishes by Lemma 2.3.3, so the claim is proven.

Since $\mathcal{M}$ is flat over Y, tensoring (2.11) with $\mathcal{M}$ and taking cohomology, we get that

$$H^0(Y, \mathcal{M}) \to H^0(Y, \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{C_i})$$

is surjective. So generic sections of $\mathcal{M}$ correspond to generic sections of $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_{C_i}$.

Let $\varphi \colon \mathcal{O}_Y^{\oplus r-1} \to \mathcal{M}$ and $\theta \colon \mathcal{O}_Y^{\oplus r+1} \to \mathcal{M}$ be defined by choosing generic sections of $\mathcal{M}$. We claim that they have maximal rank everywhere. It suffices to check the rank on closed points, so it is enough to check it on the curves C_i's. Notice moreover that, by upper semi-continuity of the rank, it is enough to check it on the unreduced points of C_i. In conclusion, we may suppose to be working on $\mathbb{P}^1$, where the claim is trivial.

Therefore $\mathrm{coker}(\varphi)$ and $\ker(\theta)$ are line bundles. Clearly, they must be $\mathcal{L}$ and $\mathcal{L}^{-1}$ respectively. □

Lemma 2.5.12. A vector bundle $\mathcal{M}$ belongs to $\mathfrak{V}$ if and only if it occurs in a short exact sequence

$$0 \to \mathcal{O}_Y^{\oplus r-1} \to \mathcal{M} \to \mathcal{L} \to 0, \tag{2.12}$$

which is determined by a set of $r-1$ generators of $H^1(Y, \mathcal{L}^{-1})$, where $\mathcal{L} \in \mathrm{Pic}^+(Y)$. $\mathcal{M}$ is uniquely determined by $\mathcal{L} = \det(\mathcal{M})$ up to addition of copies of $\mathcal{O}_Y$.

Dually, a vector bundle $\mathcal{N}$ belongs to $\mathfrak{V}^\vee$ if and only if it it occurs in a short exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}_Y^{\oplus r+1} \to \mathcal{L} \to 0,$$

which is determined by a set of $r+1$ generators of $H^0(Y, \mathcal{L})$, where $\mathcal{L} \in \mathrm{Pic}^+(Y)$. $\mathcal{N}$ is uniquely determined by $\mathcal{L} = \det(\mathcal{N})$ up to addition of copies of $\mathcal{O}_Y$.

Proof. We prove the first statement. The fact that every $\mathcal{M}$ which occurs in a short exact sequence (2.12) belongs to $\mathfrak{V}$ is proven analogously to Proposition 2.3.15.

Suppose then $\mathcal{M} \in \mathfrak{V}$. By Lemma 2.5.11, $\mathcal{M}$ occurs in

$$0 \to \mathcal{O}_Y^{\oplus r-1} \to \mathcal{M} \to \mathcal{L} \to 0,$$

where $\mathcal{L} = \det(\mathcal{M})$. First of all notice that $\mathcal{L} = \det(\mathcal{M})$ is generated by global sections, since $\mathcal{M} \in \mathfrak{V}$. In order to show that such a sequence is determined by the choice of a set of $r-1$ generators of $H^1(Y, \mathcal{L}^{-1})$, we proceed as in Proposition 2.3.15. Indeed, dualizing and taking the long exact sequence in cohomology, we get

$$H^0(Y, \mathcal{O}_Y)^{\oplus r-1} \simeq R^{r-1} \to H^1(Y, \mathcal{L}^{-1}) \to H^1(Y, \mathcal{M}^\vee) = 0,$$

proving the claim. We are left to show that $\mathcal{M}$ is determined by $\mathcal{L} = \det(\mathcal{M})$ up to addition of copies of $\mathcal{O}_Y$. Notice that any set of generators of $H^1(Y, \mathcal{L}^{-1})$ contains a minimal set of generators, and that adding extra generators corresponds to adding free summands to $\mathcal{M}$, thus concluding the proof of the first statement.

The proof of the second statement is analogous. □

Therefore, the following proposition is proven.

Proposition 2.5.13. The map

$$\begin{aligned}\varphi\colon \mathfrak{V} &\to \mathbb{Z} \times \mathrm{Pic}^{+}(Y)\\ \mathcal{M} &\mapsto (\mathrm{rk}(\mathcal{M}), \det(\mathcal{M}))\end{aligned}$$

is a group homomorphism, that is injective on isomorphism classes.

The injectivity of φ gives a way to find the indecomposable objects in $\mathfrak{V}$. Indeed, define $\mathcal{M}_0$ to be $\mathcal{O}_Y$, and for $i = 1, \ldots, n$ define $\mathcal{M}_i$ to be the extension

$$0 \to \mathcal{O}_Y^{r_i-1} \to \mathcal{M}_i \to \mathcal{L}_i \to 0 \tag{2.13}$$

associated to a minimal set of $r_i - 1$ generators of $H^1(Y, \mathcal{L}_i^{-1})$.

Proposition 2.5.14. The $\mathcal{M}_i$'s are indecomposable objects in $\mathfrak{V}$ and every indecomposable object is isomorphic to some $\mathcal{M}_i$. Moreover, for $i > 1$, the rank of $\mathcal{M}_i$ equals the multiplicity of the curve C_i in C.

Proof. The indecomposability of $\mathcal{M}_0$ is trivial, so let $i > 0$. If $\mathcal{M}_i$ admits a decomposition, it must be as $\mathcal{M}_i = \mathcal{O}_Y^{\oplus a} \oplus \mathcal{M}'$, with $\mathcal{M}'$ indecomposable such that $\det(\mathcal{M}') = \det(\mathcal{M}_i) = \mathcal{L}_i$. Clearly $r' := \mathrm{rk}(\mathcal{M}') \leq r_i$. By Lemma 2.5.12, $\mathcal{M}'$ occurs in a short exact sequence

$$0 \to \mathcal{O}_Y^{r'-1} \to \mathcal{M}' \to \mathcal{L}_i \to 0$$

determined by $r' - 1$ generators of $H^1(Y, \mathcal{L}_i)$. Therefore $r' \leq r_i$, proving $r' = r_i$ and $\mathcal{M}' = \mathcal{M}$.

We now prove that the $\mathcal{M}_i$'s are the only indecomposable objects. Using the fact that $\det(\mathcal{M}_i) = \mathcal{L}_i$ and that the $\mathcal{L}_i$'s generate $\mathrm{Pic}(Y)$, we get that for any $\mathcal{M} \in \mathfrak{V}$ there exists

$$\mathcal{R} = \mathcal{M}_1^{\oplus a_1} \oplus \cdots \oplus \mathcal{M}_n^{\oplus a_n}$$

such that $\det(\mathcal{R}) = \det(\mathcal{M})$. But then, by Lemma 2.5.12, $\mathcal{R}$ and $\mathcal{M}$ must differ by a free summand, proving that any $\mathcal{M} \in \mathfrak{V}$ can be decomposed as a direct sum of the $\mathcal{M}_i$'s.

Finally, we prove the assertion on the ranks. By Lemma 2.5.9, $\mathcal{L}_i^{-1} = \mathcal{O}_Y(-D_i)$. Consider the short exact sequence

$$0 \to \mathcal{O}_Y(-D_i) \to \mathcal{O}_Y \to \mathcal{O}_{D_i} \to 0.$$

The vanishing of $H^1(Y, \mathcal{O}_Y)$ yields the exact sequence

$$R \to H^0(Y, \mathcal{O}_{D_i}) \to H^1(Y, \mathcal{O}_Y(-D_i)) \to 0,$$

where the first morphism is actually a ring homomorphism, so its image is not contained in $\mathfrak{M}H^0(Y, \mathcal{O}_{D_i})$. This proves that $r_i - 1$, i.e. the minimal number of generators of $H^1(Y, \mathcal{O}_Y(-D_i))$, is equal to $r - 1$, where r is the minimal number of generators of $H^0(Y, \mathcal{O}_{D_i})$.
So $r_i = r$. By Nakayama's Lemma r is equal to $\dim_k(H^0(Y, \mathcal{O}_{D_i}/\mathfrak{M}\mathcal{O}_{D_i}))$. But this is the intersection number $C \cdot D_i$, which is by definition of D_i the multiplicity of C_i in C. $\square$

In Section 2.3, we introduced $\mathfrak{V}$ to study $\mathrm{Per}^{-1}(Y/X)$, and the dual $\mathfrak{V}^{\vee}$ to study $\mathrm{Per}^{0}(Y/X)$. It shouldn't therefore be surprising that, while the $\mathcal{M}_i$'s will come up in the study of $\mathrm{Per}^{-1}(Y/X)$, their duals will instead come up in the study of $\mathrm{Per}^{0}(Y/X)$.

With this in mind, define $\mathcal{N}_i := \mathcal{M}_i^{\vee}$. By Lemma 2.5.12 $\mathcal{N}_i$ occurs in a short exact sequence

$$0 \to \mathcal{N}_i \to \mathcal{O}_Y^{r_i+1} \to \mathcal{L}_i \to 0. \tag{2.14}$$

Theorem 2.5.15. The indecomposable projective objects in $\mathrm{Per}^{-1}(Y/X)$ are the $\mathcal{M}_i$'s. The projective generators in $\mathrm{Per}^{-1}(Y/X)$ are of the form $\oplus_i \mathcal{M}_i^{\oplus a_i}$ with $a_i > 0$ for all i.

Dually, the indecomposable projective objects in $\mathrm{Per}^0(Y/X)$ are the $\mathcal{N}_i$'s. The projective generators in $\mathrm{Per}^0(Y/X)$ are of the form $\oplus_i \mathcal{N}_i^{\oplus b_i}$ with $b_i > 0$ for all i.

Proof. By Prop.2.3.17, the objects in $\mathfrak{V}$ are exactly the projective objects in $\mathrm{Per}^{-1}(Y/X)$, so by Prop.2.5.14 the $\mathcal{M}_i$'s are the indecomposable projective objects in $\mathrm{Per}^{-1}(Y/X)$. Therefore, any projective object is decomposed as $\mathcal{M} = \oplus_i \mathcal{M}_i^{\oplus a_i}$. By Proposition 2.3.18, such an $\mathcal{M}$ is a projective generator if and only if $\mathcal{O}_Y$ is a direct summand of $\mathcal{M}$, giving $a_0 > 0$, and $\det(\mathcal{M})$ is ample, giving via Lemma 2.5.10 that $a_i > 0$ for $i > 0$.

The proof for $\mathrm{Per}^0(Y/X)$ is analogous. $\square$

This theorem gives a natural choice of projective generators for $\mathrm{Per}^{-1}(Y/X)$ and $\mathrm{Per}^0(Y/X)$. Indeed, we can take

$$\bigoplus_i \mathcal{M}_i \quad \text{and} \quad \bigoplus_i \mathcal{N}_i.$$

Applying Theorem 2.3.20 using these projective generators, we get the following result.

Theorem 2.5.16 ([Van04, Theorem 3.5.6])**.** There are equivalences $\mathrm{Per}^{-1}(Y/X) \simeq \mathrm{mod}(A)$ and $\mathrm{Per}^0(Y/X) \simeq \mathrm{mod}(A^{op})$, where A is a finite R-algebra such that $A/\operatorname{rad}(A) \simeq k^{n+1}$.

Proposition 2.5.17. The $n+1$ simple objects in $\mathrm{Per}^{-1}(Y/X)$ are $\mathcal{S}_0 = \mathcal{O}_C$ and $\mathcal{S}_i = \mathcal{O}_{C_i}(-1)[-1]$ for $i = 1, \dots, n$.

Proof. Notice that $\mathcal{S}_0$ is globally generated, so by Lemma 2.3.2 it belongs to $\mathcal{T}_{-1}$. For $i > 0$, notice that $f_*\mathcal{O}_{C_i}(-1)$ is supported at x, but for any open $U \subseteq X$ containing x we have

$$\Gamma(U, f_*\mathcal{O}_{C_i}(-1)) = \Gamma(f^{-1}(U) \cap C_i, \mathcal{O}_{C_i})(-1)) = \Gamma(C_i, \mathcal{O}_{C_i}(-1)) = 0,$$

so actually $f_*\mathcal{O}_{C_i}(-1) = 0$ and $\mathcal{O}_{C_i}(-1) \in \mathcal{F}_{-1}$. This proves that $\mathcal{S}_i$ belongs to $\mathrm{Per}^{-1}(Y/X)$ for all i.

To prove thesis, it suffices to show that

$$\mathrm{Hom}_{\mathrm{Per}^{-1}(Y/X)}(\mathcal{M}_i, \mathcal{S}_j) = \mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathcal{M}_i, \mathcal{S}_j) = \delta_{i,j} \cdot k.$$

Suppose $i = 0$ and $j = 0$. Then

$$\mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathcal{O}_Y, \mathcal{O}_C) = \mathrm{Hom}_Y(\mathcal{O}_Y, \mathcal{O}_C) = \mathrm{Hom}_C(\mathcal{O}_C, \mathcal{O}_C) = H^0(C, \mathcal{O}_C) = k.$$

Suppose $i = 0$ and $j > 0$. Then

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathcal{O}_Y, \mathcal{O}_{C_j}(-1)[1]) = \mathrm{Ext}^1_Y(\mathcal{O}_Y, \mathcal{O}_{C_j}) = \mathrm{Ext}^1_{C_j}(\mathcal{O}_{C_j}, \mathcal{O}_{C_j}(-1)) \\ = H^1(C_j, \mathcal{O}_{C_j}(-1)).\end{aligned}$$

By Serre duality

$$H^1(C_j, \mathcal{O}_{C_j}(-1)) = H^0(C_j, \mathcal{O}_{C_j}(-1)) = 0.$$

Suppose $i > 0$ and $j = 0$. Notice that by Lemma 2.3.3, $H^2(Y, \mathcal{M})$ vanishes for any coherent sheaf $\mathcal{M}$, and therefore $H^1(Y, -)$ is right exact on sequences of coherent sheaves. This proves that if M is a finitely generated R-module, then

$$H^1(Y, \mathcal{L}_i^{-1} \otimes_R M) = H^1(Y, \mathcal{L}_i^{-1}) \otimes_R M. \tag{2.15}$$

Indeed, M is realized as the cokernel of a homomorphism of free R-modules, and it is clear that the functors $H^1(Y, \mathcal{L}_i^{-1} \otimes_R -)$ and $H^1(Y, \mathcal{L}_i^{-1}) \otimes_R -$ coincide on free R-modules, so by right exactness they must yield the same cokernel. Applying (2.15) with $M = R/\mathfrak{M}$, we get that a minimal system of generators for $H^1(Y, \mathcal{L}_i^{-1})$ corresponds to a base over k of

$$H^1(Y, \mathcal{L}_i^{-1} \otimes_R R/\mathfrak{M}) = H^1(Y, \mathcal{L}_i^{-1} \otimes_{\mathcal{O}_Y} \mathcal{O}_C) = \mathrm{Ext}^1_Y(\mathcal{L}_i, \mathcal{O}_C).$$

Applying $\mathrm{Hom}_Y(-, \mathcal{O}_C)$ to the short exact sequence (2.13) obtained by choosing a minimal set of generators $H^1(Y, \mathcal{L}_i^{-1})$, we get

$$0 \to \mathrm{Hom}_Y(\mathcal{L}_i, \mathcal{O}_C) \to \mathrm{Hom}_Y(\mathcal{M}_i, \mathcal{O}_C) \to \mathrm{Hom}_Y(\mathcal{O}_Y^{r_i-1}, \mathcal{O}_C) \to \mathrm{Ext}^1(\mathcal{L}_i, \mathcal{O}_C).$$

By the above observations, the last morphism is actually an isomorphism, so

$$\mathrm{Hom}_Y(\mathcal{L}_i, \mathcal{O}_C) \to \mathrm{Hom}_Y(\mathcal{M}_i, \mathcal{O}_C)$$

must be an isomorphism as well. In conclusion,

$$\mathrm{Hom}_Y(\mathcal{M}_i, \mathcal{O}_C) = \mathrm{Hom}_Y(\mathcal{L}_i, \mathcal{O}_C) = \mathrm{Hom}_C(\mathcal{O}_C(D_i), \mathcal{O}_C) = H^0(C, \mathcal{O}_C(-D_i)).$$

But $H^0(C, \mathcal{O}_C(-D_i))$ is a proper ideal of $H^0(C, \mathcal{O}_C) = k$, so it must be zero.

Lastly, suppose $i > 0$ and $j > 0$. Then

$$\mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathcal{M}_i, \mathcal{O}_{C_j}(-1)) = \mathrm{Ext}^1_Y(\mathcal{M}_i, \mathcal{O}_{C_j}(-1)).$$

In the long exact sequence obtained from (2.13) by applying $\mathrm{Hom}_Y(-, \mathcal{O}_{C_j}(-1))$, there are two vanishing terms:

$$\mathrm{Hom}_Y(\mathcal{O}_Y^{r_i-1}, \mathcal{O}_{C_j}(-1)) = \mathrm{Hom}_{C_j}(\mathcal{O}_{C_j}, \mathcal{O}_{C_j}(-1))^{r_i-1} = H^0(C_j, \mathcal{O}_{C_j}(-1))^{r_i-1} = 0,$$

$$\begin{aligned}\mathrm{Ext}^1_Y(\mathcal{O}_Y^{r_i-1}, \mathcal{O}_{C_j}(-1)) = \mathrm{Ext}^1_{C_j}(\mathcal{O}_{C_j}, \mathcal{O}_{C_j}(-1))^{r_i-1} = \\ H^1(C_j, \mathcal{O}_{C_j}(-1))^{r_i-1} = H^0(C_j, \mathcal{O}_{C_j}(-1))^{r_i-1} = 0.\end{aligned}$$

So we get an isomorphism between $\mathrm{Ext}^1_Y(\mathcal{M}_i, \mathcal{O}_{C_j}(-1))$ and $\mathrm{Ext}^1_Y(\mathcal{L}_i, \mathcal{O}_{C_j}) = \mathrm{Ext}^1_{C_j}(\mathcal{L}, |C_j, \mathcal{O}_{C_j}(-1))$. For $i \neq j$, $\mathcal{L}_i|C_j$ vanishes. For $i = j$, we get

$$\mathrm{Ext}^1_{C_j}(\mathcal{L}, |C_j, \mathcal{O}_{C_j}(-1)) = \mathrm{Ext}^1_{C_j}(\mathcal{O}_{C_j}(1), \mathcal{O}_{C_j}(-1)) = H^1(C_j, \mathcal{O}_{C_j}(-2)) = H^0(C_j, \mathcal{O}_{C_j}) = k,$$

concluding the proof. □

Symmetrically, we can prove an analogous result for $\mathrm{Per}^0(Y/X)$.

Proposition 2.5.18. The $n+1$ simple objects in $\mathrm{Per}^0(Y/X)$ are $\mathcal{S}_0 = \omega_C[1]$ and $\mathcal{S}_i = \mathcal{O}_{C_i}(-1)$ for $i = 1, \ldots, n$.

2.6 Birational Geometry

In this section, we apply the equivalence proven in Section 2.4 to the study of certain birational morphisms.

First, we introduce an affine analogous of the sheaves $\mathcal{M}_i$ and $\mathcal{N}_i$ defined in (2.13). Let P be an n-dimensional complete local ring, with $n \geq 3$. Let R be a normal integral Gorenstein P-algebra, which is a free P-module of rank 2. The class group of R is the group $\mathrm{Cl}(R)$, whose elements are isomorphism classes of reflexive rank 1 R-modules, with product defined by

$$I \cdot J = (I \otimes_R J)^{\vee\vee}.$$

Let I be a reflexive R-module of depth $\geq n-1$. Notice that then

$$\mathrm{Ext}^i_R(I, R) = 0 \tag{2.16}$$

for all $i \geq 2$. Consider the following short exact sequences

$$0 \to R^{\oplus r-1} \to M \to I \to 0, \tag{2.17}$$

$$0 \to N \to R^{\oplus s+1} \to I \to 0, \tag{2.18}$$

where (2.17) is obtained by choosing a set of $r-1$ generators of $\mathrm{Ext}^1_R(I, R)$. Taking the long exact sequences obtained by applying $\mathrm{Hom}_R(-, R)$, condition (2.16) immediately implies

$$\mathrm{Ext}^i_R(M, R) = 0 \text{ for all } i \geq 2 \quad \text{and} \quad \mathrm{Ext}^i_R(N, R) = 0 \text{ for all } i \geq 1.$$

Actually, it turns out that, due to the construction of (2.17), $\mathrm{Ext}^1_R(M, R)$ vanishes as well. Indeed, the long exact sequence in degree one is

$$\mathrm{Hom}_R(R, R^{\oplus r-1}) = R^{\oplus r-1} \to \mathrm{Ext}^1_R(I, R) \to \mathrm{Ext}^1_R(M, R) \to 0.$$

By definition, the homomorphism $R^{\oplus r-1} \to \mathrm{Ext}^1_R(I, R)$ is surjective, thus implying the vanishing of $\mathrm{Ext}^1_R(M, R)$. In conclusion,

$$\mathrm{Ext}^i_R(M, R) = \mathrm{Ext}^i_R(N, R) = 0 \quad \text{for all } i \geq 1. \tag{2.19}$$

Notice that, since R is Gorenstein, R is a canonical R-module, so condition (2.19) implies that M and N are maximal Cohen-Macaulay R-modules (see [EJ00, §9.5.]). Using a global version of Lemma 2.5.12, we get that M and N are uniquely determined by I up to addition of free summands. Denote by $M(I)$ and $N(I)$ the modules obtained from M and N respectively, after deleting the free summands.

Proposition 2.6.1. In the described situation, there is an isomorphism $N(I) \simeq M(I^{-1})$.

Proof. First of all notice that, since R is free over P, for all R-modules M we have

$$\mathrm{depth}_R(M) = \mathrm{depth}_P(M).$$

Moreover, M is reflexive over R if and only if it is reflexive over P.

The module I has rank 1 over R, therefore it has rank 2 over P. Now we consider $R \otimes_P I$, which has both a natural P-module and R-module structure. In particular, as a P-module it is isomorphic to $I \oplus I$, so

$$\mathrm{depth}_P(R \otimes_P I) = \mathrm{depth}_P(I) = \mathrm{depth}_R(I) \geq n-1.$$

Moreover, since I is reflexive, it follows that $R \otimes_P I$ is reflexive over P and hence over R. Notice that

$$\mathrm{rk}_R(R \otimes_P I) = \mathrm{rk}_P(I) = 2.$$

Let K be the kernel of the natural R-modules homomorphism $R \otimes_P I \to I$, so we have the short exact sequence

$$0 \to K \to R \otimes_P I \to I \to 0.$$

As observed, $R \otimes_P I$ and I are both reflexive, so K is as well (see [Stacks, Tag 0AUY]). By additivity of the rank along short exact sequences, we get that $\mathrm{rk}_R(K) = 1$. Denote by $\det(M)$ the class of $\left(\bigwedge^{\mathrm{rk}\, M} M\right)^{\vee\vee}$ in $\mathrm{Cl}(R)$. Then determinant is multiplicative along short exact sequences and therefore

$$K \cdot I = \det(R \otimes_P I).$$

Since P is regular local, we have $\mathrm{projdim}_P(I) = 1$. But R is flat as a P-module and hence $\mathrm{projdim}_R(R \otimes_P I) = 1$. By taking a free resolution of length one and using multiplicativity of the determinant, we get

$$\det(R \otimes_P I) = R.$$

So $K = I^{-1}$ in $\mathrm{Cl}(R)$.

Consider the short exact sequence (2.18). By possibly adding free summands, we can construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & R^{\oplus s+1} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, equal] & 0 \\
 & & R \otimes_P I \arrow[r] \arrow[d] & I \arrow[r] & 0 \ . \\
 & & 0 & &
\end{tikzcd}$$

Since $\mathrm{projdim}_R(R \otimes_P I) = 1$, we can extend the diagram as follows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & R^{\oplus s-1} \arrow[r, equal] \arrow[d] & R^{\oplus s-1} \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & R^{s+1} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^{-1} \arrow[r] \arrow[d] & R \otimes_P I \arrow[r] \arrow[d] & I \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

Taking the long exact sequence obtained by applying $\mathrm{Hom}_R(-, R)$ to the left-most vertical sequence, we see, using that $\mathrm{Ext}^1_R(N, R) = 0$, that such sequence is obtained by choosing a set of generators for $\mathrm{Ext}^1_R(I^{-1}, R)$. This proves that, after deleting free summands, $N = M(I^{-1})$. Therefore $N(I) = M(I^{-1})$. $\square$

Lemma 2.6.2. Let $f\colon Y \to X$ be a projective birational morphism between normal Noetherian schemes. Suppose that the exceptional locus of f has codimension at least two in Y. Then f_* restricts to an equivalence between the category of reflexive $\mathcal{O}_Y$-modules and the category of reflexive $\mathcal{O}_X$-modules.

Proof. Without loss of generality we may assume X and Y to be integral. Let $E \subseteq Y$ be the exceptional locus of f, which by hypothesis has codimension at least 2 in Y. By [Liu02, §4.4, Corollary 4.3] $f(E)$ has codimension at least 2 in X. Let U, V be respectively $Y \setminus E$ and $X \setminus f(E)$. Since

$$f|_U\colon U \to V$$

is an isomorphism, f_* clearly defines an equivalence between the category of reflexive $\mathcal{O}_U$-modules and the category of reflexive $\mathcal{O}_V$-modules. To conclude the proof, it suffices to show that the immersions

$$i\colon U \hookrightarrow Y, \quad j\colon V \hookrightarrow X$$

restrict to equivalences on the respective categories of reflexive modules.

We consider i, the conclusion for j is clearly analogous. In particular, we show that if $\mathcal{F}$ is reflexive on Y, then $\mathcal{F}|_U$ is reflexive and $i_*(\mathcal{F}|_U)$ is naturally isomorphic to $\mathcal{F}$. Symmetrically, we show that if $\mathcal{G}$ is reflexive on U then $i_*\mathcal{G}$ is reflexive on X, while it is clear that $(i_*\mathcal{G})|_U = \mathcal{G}$.

Since U is open in Y, taking the dual commutes with restricting to U. Then it is trivial that $\mathcal{F}$ reflexive implies $\mathcal{F}|_U$ reflexive. The fact that $i_*(\mathcal{F}|_U)$ is naturally isomorphic to $\mathcal{F}$ is [Har80, Proposition 1.6].

The last thing left to prove is that if $\mathcal{G}$ is reflexive on U, then $i_*\mathcal{G}$ is reflexive on Y. Notice that $(i_*\mathcal{G})^{\vee\vee}$ is reflexive, so by the first case there is a natural isomorphism

$$(i_*\mathcal{G})^{\vee\vee} \simeq i_*((i_*\mathcal{G})^{\vee\vee}|_U).$$

As observed, restricting commutes with taking the dual, so the right-hand side is just

$$i_*((i_*\mathcal{G})|_U^{\vee\vee}) = i_*(\mathcal{G}^{\vee\vee}),$$

which is $i_*\mathcal{G}$ since $\mathcal{G}$ is reflexive. Therefore $(i_*\mathcal{G})^{\vee\vee} = i_*\mathcal{G}$, as claimed. $\square$

Let $f\colon Y \to X$ be a projective birational morphism between Noetherian schemes such that the exceptional locus of f has codimension at least 2 in Y. Suppose moreover that Y is normal Gorenstein and that $X = \operatorname{Spec}(R)$ is affine, where R is a normal complete local k-algebra with residue field k and with a canonical hypersurface singularity of multiplicity two. It follows that $\mathbf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$, so the situation falls under the case studied in Section 2.5.

Under these hypotheses, Y is integral and therefore the group $\operatorname{Cl}(Y)$ of classes of Weil divisors is isomorphic to the group of isomorphism classes of coherent reflexive $\mathcal{O}_Y$-modules of rank 1 (see [Stacks, Tag 033H, Tag 0EBK]). Then the previous lemma shows that

$$\operatorname{Cl}(Y) \simeq \operatorname{Cl}(R). \tag{2.20}$$

On the elements of $\operatorname{Pic}(Y)$, this identification maps a line bundle $\mathcal{L}$ to its global sections. Therefore, by Lemma 2.3.22, the elements of $\operatorname{Pic}(Y)$ are mapped to reflexive R-modules of depth at least $n-1$.

Consider the $\mathcal{O}_Y$-modules $\mathcal{L}_i$, $\mathcal{M}_i$ and $\mathcal{N}_i$ defined in Section 2.5. By Lemma 2.3.22, the R-modules $\Gamma(Y, \mathcal{M}_i)$ and $\Gamma(Y, \mathcal{N}_i)$ are Cohen-Macaulay. Put $I_i = \Gamma(Y, \mathcal{L}_i)$. As noticed above, the I_i's are reflexive R-modules of rank 1 and depth at least $n-1$.

For $i = 1, \dots, n$ put $M_i = M(I_i)$ and $N_i = N(I_i)$, for $i = 0$ put $M_0 = N_0 = R$. The following lemma gives a link between Section 2.5 and the construction studied in the first part of this section.

Lemma 2.6.3. We have $M_i = \Gamma(Y, \mathcal{M}_i)$ and $N_i = \Gamma(Y, \mathcal{N}_i)$ for all $i = 0, \dots, n$.

Proof. The case $i = 0$ is trivial, so we suppose $i > 0$. We study the first equality. Since $H^1(Y, \mathcal{O}_Y)$ vanishes, the long exact sequence obtained from (2.13) is

$$0 \to R^{r_i - 1} \to \Gamma(Y, \mathcal{M}_i) \to I_i \to 0. \tag{2.21}$$

Applying $\operatorname{Hom}_R(-, R)$ to (2.21) and taking the long exact Ext sequence, we see that the sequence (2.21) is obtained by choosing $r_i - 1$ generators for $\operatorname{Ext}^1_R(I_i, R)$. Therefore $\Gamma(Y, \mathcal{M}_i)$ is obtained from M_i by adding free summands. Since the equivalence (2.20) is obtained by taking global sections and the $\mathcal{M}_i$'s are indecomposable by Proposition 2.5.14, we get that the modules $\Gamma(Y, \mathcal{M}_i)$ are indecomposable. Therefore, $\Gamma(Y, \mathcal{M}_i) = M_i$.

We now study the second equality. As before, taking the cohomology sequence of (2.14) and using that $H^1(Y, \mathcal{N}_i) = 0$, as $\mathcal{N}_i \in \mathfrak{V}^\vee$, we get

$$0 \to \Gamma(Y, \mathcal{N}_i) \to R^{r_i + 1} \to I_i \to 0.$$

Therefore $\Gamma(Y, \mathcal{N}_i)$ is obtained from N_i by adding a free summand. As before, the indecomposability of $\mathcal{N}_i$ implies that of $\Gamma(Y, \mathcal{N}_i)$. So $\Gamma(Y, \mathcal{N}_i) = N_i$, concluding the proof. □

In the last part of this chapter, we apply the the proven equivalence (Theorem 2.4.2) to the study of flops.

Definition 2.6.4. Let $f \colon Y \to X$ be a projective birational morphism such that the exceptional locus of f has codimension at most two in Y and let D be an f-ample divisor on Y. We say that a projective birational morphism $f^+ \colon Y^+ \to X$ is a *flop* of f if the exceptional locus of f^+ has codimension at most two and the strict transform E of D in Y^+ is such that $-E$ is f^+-ample.

We go back to the case under study. The morphism $f \colon Y \to X$ admits a flop $f^+ \colon Y^+ \to X$. We can give an explicit construction of f^+. Since R is a complete k-algebra with a canonical hypersurface singularity of multiplicity two, we have

$$R = k[[x_1, \dots, x_{n+1}]]/(x_1^2 + f(x_2, \dots, x_n)).$$

Let $\sigma \colon X \to X$ be defined by $(x_1, x_2 \dots, x_{n+1}) \mapsto (-x_1, x_2, \dots x_{n+1})$. Then $Y^+ = Y$ and $f^+ = \sigma \circ f$, see [Kol89, Example 2.3]. To distinguish between Y and Y^+, we denote the fiber over x in the latter as C^+, the components of C^+ as C_i^+ and the divisors defined in Theorem 2.5.9 as D_i^+.

Applying (2.20) both to f and f^+ we get canonical identifications

$$\operatorname{Cl}(Y) = \operatorname{Cl}(R) = \operatorname{Cl}(Y^+).$$

In particular, $\mathcal{L}_i \in \operatorname{Pic}(Y)$ is identified with $\mathcal{L}_i^{-1} \in \operatorname{Pic}(Y^+)$. Indeed, as shown in [Kol89, Example 2.3], σ induces the endomorphism $I \mapsto I^{-1}$ on $\operatorname{Cl}(R)$. Therefore, $\mathcal{L}_i^+ = \mathcal{L}_i^{-1}$ and $I_i^+ = I_i^{-1}$.

Proposition 2.6.5. In the situation under study, $M_i^+ = N_i$ and $N_i^+ = M_i$.

Proof. It suffices to use Proposition 2.6.1. Indeed,

$$M_i^+ = M(I_i^+) = M(I_i^{-1}) = N(I_i) = N_i,$$
$$N_i^+ = N(I_i^+) = N(I_i^{-1}) = M(I_i) = M_i,$$

proving the thesis. □

Finally, we can prove more general results on flops by reduction to the formal case.

Theorem 2.6.6. Let $f\colon Y \to X$ be a projective birational morphism between normal k-varieties of dimension $n \geq 3$, such that the exceptional locus has codimension at least 2 in Y. Suppose that X has hypersurface singularities of multiplicity at most two. Then the morphism f admits a unique flop $f^+\colon Y^+ \to X$. More explicitly, there exists a unique morphism $f^+\colon Y^+ \to X$ such that the following hold.

1. Y^+ is a normal k-variety and $f^+\colon Y^+ \to X$ is a birational projective morphism. The morphisms f and f^+ have fibers of the same maximal dimension. The exceptional locus of f^+ has codimension at most 2 in Y^+. Moreover, if Y is Gorenstein, then so is Y^+.

2. Using the isomorphism proven in Lemma 2.6.2, we get identifications
$$\mathrm{Cl}(Y) \simeq \mathrm{Cl}(X) \simeq \mathrm{Cl}(Y^+). \tag{2.22}$$
In particular, (2.22) restricts to an isomorphism between $\mathrm{Pic}(Y)$ and $\mathrm{Pic}(Y^+)$.

3. If E is an f-nef (resp. f-ample) divisor on Y and E^+ is its strict transform on Y^+, then $-E^+$ is f^+-nef (resp. f^+-ample).

Proof. Let D an f-ample Cartier divisor on Y. Via f we can identify it with a Weil divisor on X, which we still call D. Notice that, by [Kol89, Def 2.1], if the flop f^+ exists then it is determined uniquely by D, therefore it is unique. So we only have to prove existence. Moreover, by [KM98, Corollary 6.7], it suffices to prove existence locally, so we may suppose X to be affine.

By [KM98, Corollary 6.4.(b)], the flop f^+ exists if and only if the sheaf $\mathcal{S} = \bigoplus_n \mathcal{O}_X(-nD)$ is a sheaf of finitely generated $\mathcal{O}_X$-algebras. In such a case, $Y^+ = \underline{\mathrm{Proj}}\,\mathcal{S}$. According to [KM98, Proposition 6.6], it suffices to check this condition on the completion of the closed points of X. Therefore, we can restrict the study to the formal case $X = \mathrm{Spec}(R)$, where R is a complete k-algebra with hypersurface singularities of multiplicity 1 or 2.

By the construction given above, in this case the flop Y^+ exists and we have an explicit expression. In particular, it is easy to check that the conditions 1, 2 and 3 are verified. Since the proprieties hold when passing to the completion of the closed points, they hold in the general case as well, so the flop exists and has the desired properties. □

Theorem 2.6.7. Let $f\colon Y \to X$ be a projective birational morphism between normal, quasi-projective, Gorenstein k-varieties dimension $n \geq 3$. Suppose that the fibers of f have dimension at most 1 and that the exceptional locus of f has codimension at least 2 in Y. Suppose moreover that X has canonical hypersurface singularities of multiplicity at most 2. Let $f^+\colon Y^+ \to X$ be the flop of f. Then, there is an equivalence of triangulated categories

$$D^b(\mathrm{coh}(Y)) \simeq D^b(\mathrm{coh}(Y^+)),$$

which restricts to an equivalence

$$\mathrm{Per}^{-1}(Y/X) \simeq \mathrm{Per}^0(Y^+/X).$$

Proof. According to Proposition 2.4.3, there exists a vector bundle $\mathcal{P}$ on Y which is a local projective generator for $\mathrm{Per}^{-1}(Y/X)$. By Lemma 2.3.22, $f_*\mathcal{P}$ is Cohen-Macaulay on X and hence reflexive, see [BH98, Proposition 1.4.1].

Therefore, applying Lemma 2.6.2 to f^+, we get that $f_*\mathcal{P}$ corresponds to a reflexive $\mathcal{O}_{Y^+}$-module $\mathcal{Q}^+$.

We now want to show that $\mathcal{Q}^+$ is a local projective generator for $\mathrm{Per}^0(Y^+/X)$. Clearly, it suffices to show it by restricting to an affine cover of X, so we may suppose $X = \mathrm{Spec}(R)$. In particular, we may suppose to be in the instance of the formal case discussed in the first part of the chapter. Then, by Theorem 2.5.15, we have

$$\mathcal{P} = \bigoplus_i \mathcal{M}_i^{\oplus a_i}.$$

Lemma 2.6.3 then yields $f_*\mathcal{P} = \bigoplus_i M_i^{\oplus a_i}$. By Lemma 2.6.5, this coincides with $\bigoplus_i (N_i^+)^{\oplus a_i}$. Using now Lemma 2.6.3 with f^+, we then get that

$$\mathcal{Q}^+ = \bigoplus_i (\mathcal{N}_i^+)^{\oplus a_i}.$$

Therefore, Theorem 2.5.15 proves that $\mathcal{Q}^+$ is a projective generator for $\mathrm{Per}^0(Y^+/X)$.

Going back to the general case, we want to use the local projective generators $\mathcal{P}$ and $\mathcal{Q}^+$ in the equivalences proven in Theorem 2.4.2. The choice of $\mathcal{Q}^+$ yields

$$f_*\mathcal{E}nd_Y(\mathcal{P}) = f_*^+\mathcal{E}nd_{Y^+}(\mathcal{Q}^+).$$

We denote this sheaf of $\mathcal{O}_X$-algebras by $\mathcal{A}$. Applying Theorem 2.4.2 both to f and f^+, we get a chain of equivalences

$$D^b\left(\mathrm{coh}(Y)\right) \xrightarrow{\mathbf{R}f_*\mathbf{R}\mathcal{H}om_Y(\mathcal{P},-)} D^b\left(\mathrm{coh}(\mathcal{A})\right) \xrightarrow{(f^+)^{-1}(-)\overset{\mathbf{L}}{\otimes}_{(f^+)^{-1}(\mathcal{A})}\mathcal{Q}^+} D^b\left(\mathrm{coh}(Y^+)\right),$$

which restrict to equivalences

$$\mathrm{Per}^{-1}(Y/X) \to \mathrm{coh}(\mathcal{A}) \to \mathrm{Per}^0(Y^+/X).$$

This concludes the proof. $\square$

Appendix A

Morita Theory

We recall the basic results concerning Morita Theory of finite dimensional algebras. Two k-algebras A and B are said to be *Morita equivalent* if their module categories $\operatorname{mod}(A)$ and $\operatorname{mod}(B)$ are equivalent. Therefore, the key idea behind Morita Theory is to study not the algebras themselves, but rather their module categories.

We follow [AC20]. Let A be a finite dimensional k-algebra. In what follows, by an *A-module* we mean a finitely generated (or equivalently finite dimensional) right A-module.

Definition A.0.1. The *radical* $\operatorname{rad}(A)$ of the ring A is the ideal obtained as the intersection of all the maximal right ideals of A. Similarly, the radical $\operatorname{rad}(M)$ of an A-module M is the submodule obtained as the intersection of all the maximal submodules of M.

The radical of A has several useful properties.

Proposition A.0.2 ([Bar15, §3.6; AC20, §I.1.2; Lam91, §4])**.** The following properties hold for $\operatorname{rad}(A)$:

(a) The radical is a two-sided ideal, characterized as follows

$$\begin{aligned} \operatorname{rad}(a) &= \{a \in A \mid 1 - ax \text{ is right invertible for all } x \in A\} \\ &= \{a \in A \mid 1 - xa \text{ is left invertible for all } x \in A\}. \end{aligned}$$

(b) The radical is a nilpotent ideal, that is $\operatorname{rad}(A)^n = 0$ for n big enough.

(c) The quotient $A/\operatorname{rad}(A)$ is a semisimple algebra.

(d) An element $a \in A$ belongs to $\operatorname{rad}(A)$ if and only if it annihilates every simple A-module.

Lemma A.0.3. If M is an A-module, then $\operatorname{rad}(M) = M \cdot \operatorname{rad}(A)$.

Proof. We prove the two inclusions. Let $N \subsetneq M$ be a maximal submodule. Then the quotient M/N is a simple A-module, so by Proposition A.0.2(d) it is annihilated by $\operatorname{rad}(A)$. This proves that $M\operatorname{rad}(A) \subsetneq N$ for all maximal submodules N, hence $M\operatorname{rad}(A) \subseteq \operatorname{rad}(M)$.

By Proposition A.0.2(c) the algebra $A/\operatorname{rad}(A)$ is semisimple. Then by [Rot09, Proposition 4.5] all $A/\operatorname{rad}(A)$-modules are semisimple. In particular $M/M\operatorname{rad}(A)$ is a semisimple $A/\operatorname{rad}(A)$-module, and hence it is a semisimple A-module, see [Lam91, Prop 4.8]. This proves that $\operatorname{rad}(M) \subseteq M\operatorname{rad}(A)$. □

We now study the indecomposable A-modules. We will focus in particular on the indecomposable projective A-modules. Clearly, these are the modules which appear in a decomposition of A_A, i.e. the ring A seen as a right A-module.

Proposition A.0.4 ([AC20, Prop I.1.13])**.** An A-module M is indecomposable if and only if its endomorphism algebra $\mathrm{End}_A(M)$ is local.

Theorem A.0.5 ([AC20, Theorem I.1.14])**.** An A-module M admits a decomposition

$$M = M_1 \oplus \cdots \oplus M_n,$$

where the M_i's are indecomposable. Such a decomposition is unique, up to isomorphism and reordering.

Lemma A.0.6. Let $B_1, \dots B_n \subseteq A$ be right ideals. Then A admits a decomposition $A = B_1 \oplus \cdots \oplus B_n$ as a right A-module if and only if there exist $e_1, \dots e_n \in A$ orthogonal idempotents such that $1 = e_1 + \cdots + e_n$ and $B_i = e_i A$.

Proof. Suppose A decomposes as $A = B_1 \oplus \cdots \oplus B_n$. Therefore, there exist $e_i \in B_i$ such that $1 = e_1 + \cdots + e_n$. Then

$$e_i = 1 \cdot e_i = \sum_{j=1}^{n} e_j \cdot e_i.$$

Notice that $e_j \cdot e_i \in B_j \cdot e_i \subseteq B_j$. Since the direct sum yields a unique decomposition, we get

$$e_i^2 = e_i \quad \text{and} \quad e_j \cdot e_i = 0 \text{ for } i \neq j.$$

We are left to show that $B_i = e_i A$. Since $e_i \in B_i$ and B_i is a right ideal, the inclusion $e_i A \subseteq B_i$ is trivial. For the other inclusion, consider $b \in B_i$. Then

$$b = 1 \cdot b = \sum_{j=1}^{n} e_j \cdot b.$$

Once again, by uniqueness of the decomposition we get

$$e_i \cdot b = b \quad \text{and} \quad e_j \cdot b = 0 \text{ if } j \neq i.$$

This proves $B \subseteq e_i A$.

Suppose now that there exist $e_1, \dots e_n \in A$ with the stated properties and set $B_i = e_i A$. We need to show that $A = B_1 + \cdots + B_n$ and that $B_i \cap B_j = \{0\}$ if $i \neq j$. For the first equality, it suffices to notice that, for all $a \in A$,

$$a = 1 \cdot a = \sum_{1=0}^{n} e_i \cdot a.$$

For the second equality, let $x \in B_i \cap B_j$. Then there exist $a, b \in A$ such that

$$x = e_i \cdot a = e_j \cdot b.$$

Therefore,

$$x = e_i \cdot a = e_i^2 \cdot a = e_i \cdot x = e_i e_j \cdot b = 0 \cdot b = 0.$$

This concludes the proof. $\square$

The idempotents e_i's appearing in the previous lemma are *orthogonal*, meaning $e_ie_j = \delta_{i,j}$, and *complete*, meaning $1 = e_1 + \dots e_n$. The B_i's are indecomposable if and only if the e_i's are also *primitive*, meaning that any decomposition $e_i = e_i' + e_i''$ with e_i' and e_i'' idempotent yields $e_i' = 0$ or $e_i'' = 0$, see [AC20, §I.1.3; Bar15, Corollary 4.18]. Therefore, a set $\{e_1, \dots e_n\}$ of complete primitive orthogonal idempotents yields a decomposition

$$A_A = P_1 \oplus \cdots \oplus P_n, \tag{A.1}$$

where $P_i = e_iA$ and the P_i's are indecomposable projectives. Moreover, any indecomposable projective A-module is isomorphic to P_i for some i.

Lemma A.0.7 ([Bar15, Example 4.32])**.** The quotient $P_i / \operatorname{rad} P_i$ is a simple A-module.

Proposition A.0.8. Let M be an A-module and $e \in A$ an idempotent element. Then $\operatorname{Hom}_A(eA, M) = Me$.

Proof. A morphism $f\colon eA \to M$ is uniquely determined by $f(e) = m$. Notice that

$$m = f(e) = f(e \cdot e) = f(e) \cdot e = m \cdot e. \tag{A.2}$$

This proves that m belongs to Me. On the other hand, let $m = n \cdot e \in Me$. Then m satisfies the property (A.2) and thus defines a homomorphism $eA \to M$. Indeed,

$$m = n \cdot e = n \cdot e^2 = m \cdot e.$$

This concludes the proof. $\square$

Corollary A.0.9. If $e_x, e_y \in A$ are idempotent elements, then $e_xAe_y = \operatorname{Hom}_R(e_yA, e_xA)$.

Definition A.0.10. The ring A is *basic* if in (A.1) the P_i's are not isomorphic, i.e. $P_i \not\simeq P_j$ if $i \neq j$.

Remark A.0.11 ([AC20, §I.2.2])**.** If A is basic, then $A/\operatorname{rad}(A)$ is a product (possibly noncommutative) of fields. In particular, if the field k is algebraically closed, then $A/\operatorname{rad}(A) \simeq k^n$, where n is the number of indecomposable modules appearing in the decomposition of A.

The property highlighted by the previous remark proves to be useful.

Definition A.0.12. A k-algebra A is *elementary* if $A/\operatorname{rad}(A) \simeq k^n$.

Remark A.0.11 can be then rephrased as follows.

Proposition A.0.13. A basic algebra over an algebraically closed field is elementary.

As we will see later, a class of algebras of great importance is formed by the bounded quiver algebras. We give the necessary definitions.

Definition A.0.14. A *quiver* is an oriented graph. Formally, a quiver Q constitutes of a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of *vertices*, Q_1 is the set of *arrows*, s and t are functions $Q_1 \to Q_0$ determining the *source* and the *target* of each arrow.
A *path* on Q is a finite sequence of composable arrows of Q. Formally, a path is a sequence

$$(y|\alpha_n|\alpha_{n-1}|\dots|\alpha_1|x),$$

where $x, y \in Q_0$ are vertices and $\alpha_1, \dots, \alpha_n \in Q_1$ are arrows such that $s(\alpha_1) = x,\ t(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \dots n-1$ and $t(\alpha_n) = y$.

Notice that, under the above definition, we also have for each vertex $x \in Q_0$ the *lazy path*

$$(x|x).$$

It is the neutral element with respect to the obvious composition of paths.

Definition A.0.15. The *path algebra* kQ of the quiver Q is the algebra generated by the paths on the quiver Q. The product of two paths is defined as their composition if possible, or it is zero otherwise.

In kQ, let kQ^+ be the ideal generated by the arrows.

Definition A.0.16. An ideal $I \subseteq kQ$ is *admissible* if there exists $m \geq 2$ such that

$$(kQ^+)^m \subseteq I \subseteq (kQ^+)^2.$$

The couple (Q, I) is called a *bound quiver* and the algebra $A = kQ/I$ is called a *bound quiver algebra*.

Proposition A.0.17 ([AC20, Prop I.2.7.]). Let Q be a finite connected quiver and $I \subseteq kQ$ be an admissible ideal. Then the algebra kQ/I is a basic connected finite dimensional algebra.

We have thus seen how to construct a finitely dimensional algebra starting from a finite quiver. Symmetrically, we can construct a quiver associated to a k-algebra A.

Definition A.0.18. Let A be an elementary algebra and let $\{e_1, \dots, e_n\}$ be a set a of primitive complete orthogonal idempotents in A. The *ordinary quiver* Q_A of A is defined as follows.

- The vertices of Q_A are $\{e_1, \dots, e_n\}$.
- The number of arrows going from e_i to e_j equals the dimension over k of the ideal

$$e_i \left(\frac{\mathrm{rad}(A)}{\mathrm{rad}^2(A)} \right) e_j.$$

Lemma A.0.19. [AC20, Lemma I.2.10.] The quiver Q_A does not depend on choice of the primitive complete orthogonal idempotents $\{e_1, \dots, e_n\}$.

The link between bound quiver algebras and their ordinary quivers is given by the following results.

Lemma A.0.20. [AC20, Lemma I.2.11.] If $A = kQ/I$ is a bound quiver algebra, then $Q_A = Q$.

Theorem A.0.21. [AC20, Theorem I.2.13.] If A is an elementary finite dimensional k-algebra, then there exists an admissible ideal $I \subseteq kQ_A$ such that $A \simeq kQ_A/I$. Namely, every elementary finite dimensional algebra is a bound quiver algebra.

A way to encode the structure of the category mod A in a quiver is via the *Auslader-Reiten quiver*. Before defining it, we mention the notion of *radical morphism* and list some properties.

Definition A.0.22. If M and N are A-modules, the *radical* $\operatorname{rad}_A(M,N)$ is the set of all the homomorphisms $f \in \operatorname{Hom}_A(M,N)$ such that for any section $s\colon M' \to N$ and retraction $r\colon N \to N'$ the composition $rfs\colon M' \to N'$ is not an isomorphism.

Remark A.0.23. [AC20, Lemma II.1.6.] The set of all radical morphism form an *ideal* in the category $\operatorname{mod} A$, meaning that it is closed under left and right composition with arbitrary homomorphisms. Therefore, for all $n > 0$ there is a well defined ideal $\operatorname{rad}^n_A(M,N)$, generated by morphisms which are compositions of n radical morphisms.

Remark A.0.24. [AC20, Corollary II.1.8] If M and N are indecomposable A-modules, then $\operatorname{rad}_A(M,N)$ is the set of all the homomorphisms in $\operatorname{Hom}_A(M,N)$ which are not isomorphisms.

Lemma A.0.25. [AC20, Cor II.1.10.] Let $f\colon M \to N$ be a homomorphism of A-modules.

(a) If M is indecomposable, then f is radical if and only if it is not a section.

(b) If N is indecomposable, then f is radical if and only if it is not a retraction.

Lemma A.0.26. Let $P = eA$ be an indecomposable projective A-module and M an A-module. Then $\operatorname{rad}_A(M,P) = \operatorname{Hom}_A(M, \operatorname{rad}(P))$.

Proof. Let $S := P/\operatorname{rad} P$, which is a simple module by Lemma A.0.7, and let $\pi\colon P \to S$ be the natural projection. By simplicity of S, a morphism $g\colon M \to P$ has the property that $\pi \circ g$ is nonzero if and only if $\pi \circ g$ is surjective, hence if and only if the class of e belongs to the image of $\pi \circ g$. This means that there exist $m \in M$ and $r \in \operatorname{rad} P = e\operatorname{rad}(A)$ such that $g(m) = e - r$. Then

$$g(m + mr) = e - r + (e - r)r = e - r^2.$$

Iterating, we get that $g(m + mr + \cdots + mr^{n-1}) = e - r^n$. By Proposition A.0.2(b) the radical $\operatorname{rad}(A)$ is a nilpotent ideal, so for n big enough $r^n \in \operatorname{rad}(A)^n$ vanishes. This proves that there exists $m' \in M$ such that

$$g(m') = e.$$

Then g must be surjective. In conclusion, we showed that g is surjective if and only if its image is not contained in $\operatorname{rad} P$. On the other hand, since P is projective, g is surjective if and only if it is a retraction. Then Lemma A.0.25 shows that the radical morphisms are exactly those which factor through $\operatorname{rad} P$. $\square$

Definition A.0.27. The *space of irreducible morphisms* from M to N is

$$\operatorname{Irr}_A(M,N) = \frac{\operatorname{rad}_A(M,N)}{\operatorname{rad}^2_A(M,N)}.$$

Definition A.0.28. The *Auslander-Reiten quiver* $\Gamma(\operatorname{mod} A)$ associated to the algebra A is defined as follows.

- The vertices of $\Gamma(\operatorname{mod} A)$ are the isomorphism classes of indecomposable A-modules.

- If M and N are nonisomorphic indecomposable A-modules, the number of arrows going from $[M]$ to $[N]$ equals the dimension over k of $\operatorname{Irr}_A(M,N)$.

The Auslander-Reiten quiver is relevant, because it encodes in combinatorially much of the information about the category mod A. It can be constructed in grater generality form a k-linear category. Many of the results carry out to the more general setting.

Definition A.0.29. An algebra A is *representation-finite* if the category mod A admits a finite number of isomorphism classes of indecomposable objects.

Suppose that A is a representation-finite algebra and let $M_1, \ldots M_n$ be representatives of the isomorphism classes of indecomposable A-modules. Set

$$M = M_1 \oplus \cdots \oplus M_n.$$

Then M is an *additive generator* of mod A. Namely, mod A = add M is the smallest full subcategory of mod A containing all direct sums of direct summands of M.

Definition A.0.30. The *Auslander algebra* of A is $B := \mathrm{End}_A(M)$.

Theorem A.0.31. [AC20, Theorem VI.3.2.] Suppose that A is a representation-finite algebra and B is the Auslander algebra of A. Then the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A is isomorphic to the ordinary quiver Q_B of B.

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