Hyperbolic volume of complements of positive braids

Antonio De Capua

Merton College

University of Oxford

A report submitted for the Transfer of Status from PRS to DPhil

Trinity 2013

Table of Contents

Last edit on 13th December 2013

Introduction

Among the most important contributions of William Thurston to low-dimensional topology, in particular in knot theory and hyperbolic geometry, is that each knot in S^3 is one and only one among a torus knot, a satellite knot, or a *hyperbolic knot*, namely a link whose complement admits a complete hyperbolic structure. In other words, hyperbolic knots and, more generally, links, rather than being sort of a 'special' case, represent in some sense the widest class; and also, the one of which we understand less.

A very common issue when investigating the geometric structures on 3-manifold is that the theorems that prove hyperbolicity do not give a sufficiently explicit description of the hyperbolic structure (the *only* possible one, thanks to Mostow's Rigidity Theorem). Also, operations that are simple to describe from a topological point of view, like Dehn surgeries, do not result in a 'simple' change in the hyperbolic structure.

Our concern, in particular, is finding estimates for the *volume* of hyperbolic manifolds describable as link complements. Theoretically, a way of finding out the volume of a hyperbolic link complement would be triangulating it with ideal tetrahedra (i.e. tetrahedra without vertices: these ideally would lie "at the infinity", in the place the link we have removed used to be), and then looking for a solution for Thurston's gluing consistency equations. But a strict application of this procedure does not give a satisfactory answer to our question: our aim is looking for looser information about hyperbolic volume, but some that is valid for entire classes of knots and links rather than a particular one, and is describable in terms of properties and quantities of link diagrams.

A successful attempt in this sense is the one of Marc Lackenby (2001), which concerns *prime alternating diagrams* of links, namely diagrams that cannot be split into trivial tangles by cutting in 2 points, and such that, when running along a link component, one passes alternatively under and above the crossings of the diagram.

Lackenby found that the hyperbolic volume of the complement is linearly bounded, from both below and above, in terms of the *twist number* of the diagram (namely the number of 'rows of consecutive crossings' that it is possible to recognize in the diagram). His result is based on a previous theorem of Ian Agol according to which a lower bound on hyperbolic volume of a 3-manifold is attained by cutting it along an *incompressible* surface (i.e. one that is embedded in a π_1-injective way), and then computing the Euler characteristic of the *guts* of the resulting manifold, namely the 'pieces' of it that are not bundles over a surface with fibre the interval nor *Seifert-fibred*, i.e. bundles over an orbifold with fibre S^1. Lackenby cuts the link complement along a surface that, thanks to the alternativity of the diagram, is incompressible, and then examines the 'pieces' of the new manifold.

Here, we focus instead on another way of representing links, namely as closure of *braids*. In particular we focus on *positive* braids, namely the ones in which the crossings are all oriented on the same way. When a hyperbolic link admits a positive braid as a diagram, this diagram is likely to give essential information about the hyperbolic structure, because, even if the representation as a positive braid is not unique, it is to some extent a preferred one: it is one that cannot get simplified in any trivial way. It is already known that some properties of the link can be read off from a positive braid diagram. In particular: can one read off the hyperbolic volume of a positive braid from its positive diagram?

This work is structured as follows.

Chapter 1 is a summary of the background notions. I will talk about how a 3-manifold is split into the 'pieces' mentioned above, and how Thurston's Geometrization ensures that each of them has only one geometrical structure. Then I will list some common techniques and well-known facts about 3-manifolds: Dehn surgery, Gromov norm, criteria for hyperbolicity. Finally I introduce positive braids and a canonical decomposition for them.

Chapter 2 gravitates around a theorem by Jeffrey Brock which gives bounds for the volume of a hyperbolic 3-manifold which fibers over S^1 with a surface as a fiber (and there are at least two ways to describe the complement of a positive braid as such). The bound will be in terms of the translation distance induced by the *monodromy homeomorphism* on the *pants graph* of the fiber surface. I will describe my attempts and ideas to use Brock's result in the setting of positive braids: I will highlight in particular an approach involving *train tracks* as a means for connecting diagrammatic quantities of a braid with properties of the monodromy homeomorphism.

Chapter 3 gathers further ideas for investigation of the question. I will describe the results of Agol and Lackenby, and how they inspired a work of Futer-Kalfagianni-Purcell that aims to give technical means for applying them to a much broader variety of hyperbolic 3-manifolds. They also prove a volume bound for a restricted class of positive braids. I will then expose an idea for relating the volume to the number of distinct factors appearing in the canonical decomposition of a positive braid.

Chapter 1

Background

Before getting started, let's fix once and for all the exact meaning of some terms I am going to use. By 3-*manifold* in this work, I will always mean, unless explicitly stated otherwise, an oriented, connected, differentiable 3-manifold, possibly non-compact, without boundary unless otherwise specified, and always of *finite type*, i.e. homeomorphic to a compact 3-manifold with boundary, possibly with part of its boundary removed. By *surface* I will always mean a 2-manifold which is connected, differentiable, possibly non-compact, without boundary unless otherwise specified, and of finite type. It is allowed to be non-orientable, even without explicit mention.

1.1 Decomposition and geometrization of 3-manifolds

1.1.1 Geometric structures

This report is concerned with investigating *hyperbolic metrics* on selected 3-manifolds, which are a particular case of *geometric structure*:

Definition 1.1.1. A *3-dimensional geometry* is a pair (X, G), where X is a simply connected 3-manifold, and G is a group of diffeomorphisms of X that acts effectively and transitively. Moreover, we require that, for each $x \in X$, its stabilizer $G_x \subseteq G$ is compact. A geometry (X, G) is *maximal* if there are no geometries (X, G') with $G \subsetneq G'$.

For M a 3-manifold, a (X, G)-*structure* on M is an atlas of charts $\phi_i : U_i \to X$ where $U_i \subseteq M$ is an open subsets, such that each change of chart $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is the restriction of an element of G. This atlas is also required to cover the whole M, and to be maximal.

This definition of structure does not make any reference to Riemannian metrics, but they can be used as well for defining an (almost) equivalent notion of geometric structure. The reason why the definition above is preferred is that is does not distinguish structures that differ from each other in a trivial way (e.g. two Euclidean metrics that are the same up to rescaling). For more details see [Bon02]. Still, it makes sense to talk about a *complete* (X, G)-structure, namely one such that any of the corresponding Riemannian structures is complete as a metric space.

There are eight different maximal geometries; the *hyperbolic* one, in particular, is the one where $X = \mathbb{H}^3$ and $G = \mathrm{Isom}^+(\mathbb{H}^3)$. The meaning of *hyperbolic structure* on a 3-manifold is defined accordingly. Of course an analogous definition holds in any dimension ≥ 2.

1.1.2 Embedded surfaces

A common way of analysing the topology of a 3-manifold is focusing on the surfaces which are embedded in it in non-trivial ways. The first thing one might look at are *essential spheres*, namely 2-spheres which are embedded in the 3-manifold and do not bound a 3-ball. A 3-manifold is called *irreducible* if it does not contain any essential sphere.

If one removes an open 3-ball from two 3-manifolds M_1 and M_2, and then glues them along their newly created boundary components in such a way that the result is an oriented manifold M we say that M is the *connected sum* of M_1 and M_2 ($M = M_1 \# M_2$). It is well-defined as, up to diffeomorphisms, it does not depend on the choice of the 3-ball. If the only way one can write M as a connected sum is taking one of the summands to be S^3, it is called *prime*. A prime manifold is also irreducible. The converse is not true in general, but the only (oriented) counterexample is $S^2 \times S^1$.

Theorem 1.1.2 (Kneser). *Every 3-manifold (possibly with boundary) admits a decomposition into a connected sum of prime manifolds $M = M_1 \# \ldots \# M_n$. The M_i's are uniquely determined up to adding trivial S^3 summands.*

Anyway, it is worth noting that the summands are uniquely determined, but the collection of 2-spheres in M that gives this decomposition is not.

After examination of interesting spheres, one can move to general surfaces:

Definition 1.1.3. Let M be a 3-manifold, and $S \subset M$ be a properly embedded surface (i.e. a connected 2-manifold, not necessarily orientable, of finite type), possibly with boundary. The surface S is said to be *incompressible* if, for each embedded 2-disc $D \subset M$ with $D \cap S = \partial D$, ∂D bounds also a disc in S.

A surface S, possibly with boundary, embedded in a 3-manifold M with boundary in such a way that $\partial S = S \cap \partial M$, is *boundary-incompressible* (abbreviated as ∂-*incompressible*) if there is no disc D embedded in M such that ∂D consists of an arc properly embedded into S, and essential in it, and the other one lying on ∂M.

An irreducible 3-manifold M is *Haken* if it is has no boundary and there exists an embedded incompressible surface in M; or, if it has some boundary and there exists some incompressible and ∂-incompressible surface properly embedded in M.

Definition 1.1.4. Let S be a surface, possibly with boundary. A S-*end* of a 3-manifold M, possibly with boundary as well, is a proper embedding $S \times [0, +\infty) \hookrightarrow M$.

Definition 1.1.5. A properly embedded surface $S \hookrightarrow M$ is *essential* if its double $DS = \partial\mathcal{N}(S)$ in M is incompressible, boundary-incompressible and not homotopic into an end of M.

Reference to the double (according to the definition given in [FKP11]) is necessary in case of non-orientable, one-sided surfaces, for which the request on the double is stronger than on the surface itself.

1.1.3 JSJ decomposition

After a manifold has been split into its prime summands, one can put apart the copies of $S^2 \times S^1$ and focus on the other ones, which are also irreducible. Irreducible manifolds admit a further decomposition, as Jaco, Shalen [JS78] and Johannson [Joh79] obtained by cutting along essential tori and annuli, such that the remaining pieces are quite standard:

Definition 1.1.6. A 3-manifold (with or without boundary) is *atoroidal* if it does not contain any essential embedded torus. A 3-manifold with boundary is *anannular* if it does not contain any essential properly embedded annulus.

Definition 1.1.7. A *Seifert fibration* of a 3-manifold M is a fibration in circles such that:

- there exist a finite number of fibers $F_1, \dots, F_n$ such that each of them has an open neighbourhood U_i which is union of fibers and is diffeomorphic, as a fibered manifold, to $\mathrm{int}(D^2) \times I/\sim$ vertically fibered, where $\sim$ is the gluing of $\mathrm{int}(D^2) \times \{0\}$ with $\mathrm{int}(D^2) \times \{1\}$ after rotation by $2\pi p/q$, where p, q are coprime integers;
- any other fiber F has a neigbourhood U which is union of fibers, and the fibration is trivial.

For 3-manifolds *without boundary*, a collection of tori is enough. An essential torus T embedded in a 3-manifold M is called *canonical* if, for any other essential torus T' embedded in M, T and T' can be isotoped to be disjoint.

Theorem 1.1.8 (JSJ decomposition for 3-manifolds without boundary). *Let M be an irreducible 3-manifold, and let T be the collection of all the canonical tori in M. Then T can be considered to be a collection of pairwise disjoint, essential tori, up to isotopy.*
Moreover, each component of $M - T$ either is atoroidal or admits a Seifert fibration.
Up to isotopies, T is the unique minimal collection of pairwise disjoint, essential tori capable of performing such a splitting.
Each essential embedded torus in M is either isotopic to a canonical torus of M, or can be isotoped into a Seifert-fibered component of $M - T$, in such a way that it is either an union of fibers or transverse to each fiber it intersects.

The case of 3-manifolds with boundary requires that the boundary is incompressible (this requires a special definition):

Definition 1.1.9. The boundary of a 3-manifold is *incompressible* if, for any properly embedded disk D in M with $\partial D = D \cap \partial M$, ∂D bounds a disk in ∂M.

The JSJ decomposition this time cuts along a collection of *canonical* tori and annuli, meaning that each of them is essential, and up to isotopies that keep the boundary of each annulus on the boundary of the 3-manifold, it is disjoint to all other essential annuli and tori.

Theorem 1.1.10 (JSJ decomposition for 3-manifolds with boundary). *Let M be an irreducible 3-manifold with boundary. Suppose the boundary of M is incompressible and none of its components is a 2-sphere or a projective plane. Let F be* *the collection of all the canonical tori and annuli in M. Then F can be considered to be a collection of pairwise disjoint, essential tori and annuli, up to isotopy. Moreover, each component W of $M - F$ has one of the following properties:*

- *it is atoroidal and anannular;*
- *it is Seifert-fibered, in such a way that $W \cap \partial M$ is an union of fibers;*
- *it is an I-bundle over a surface, in such a way that the $\{0, 1\}$ sub-bundle coincides with $W \cap \partial M$.*

Up to isotopies, F is the unique minimal collection of pairwise disjoint, essential tori and annuli capable of performing such a splitting.
Each essential annulus, up to isotopies that keep its boundary on ∂M, is one of the following:

- *an element of F;*
- *contained in a Seifert-fibered component of $M - F$, in such a way that it is an union of fibers;*
- *contained in an I-bundle component of $M - F$, in such a way that all of its radii are fibers.*

Definition 1.1.11. The *characteristic submanifold* of a 3-manifold M is the union of all the Seifert-fibered and I-bundles components in the JSJ decomposition of Theorems 1.1.8 and 1.1.10. The *guts* of a 3-manifold M is the union of all the atoroidal components.

1.1.4 The geometrization theorem

A famous conjecture of Thurston, recently proved by Perelman (see [CZ06]), states that each piece in the JSJ decomposition of a 3-manifold admits an unique geometric structure up to trivial changes. When the manifold has boundary, it is required that the geometric structure is rigid on its boundary:

Definition 1.1.12. Let W be a 3-manifold with boundary, and let DW be its double, namely the disjoint union of W with a copy W^- of W with the opposite orientation, where we identify each point on ∂W with the corresponding point on ∂W^-. A *geometric structure with totally geodesic boundary* on W is a geometric structure on DW such that each point x in the image of ∂W has a neighourhood U_x with the property that any shortest geodesic arc in DW connecting two points of U_x entirely lies in U_x.

An (X, G)-structure on W with totally geodesic boundary is a (X, G)-structure on DW such that any geometric structure which induces it is a totally geodesic structure on W as described above.

Theorem 1.1.13 (Thurston's Geometrization). *Let M be a 3-manifold, possibly with boundary. In this case, suppose the boundary of M is incompressible and none of its components is a 2-sphere or a projective plane. Then each component of $M - T$ in Theorem 1.1.8, or $M - F$ in Theorem 1.1.10, admits a complete geometric structure, with totally geodesic boundary if any. If all the ends of M are toric ends or annular ends, these geometric structures can be chosen to have finite volume, except in the following cases:*

- *M is non-compact and is an (open, closed or half-open) interval bundle over a plane, an open annulus, an open Möbius band, a torus or a Klein bottle;*
- *T or F consists of a single torus, and the complement is diffeomorphic to $S^1 \times S^1 \times \mathbb{R}$.*

Each component might admit more than one geometric structure, but all of them are modelled on the same geometry.

1.2 Facts about hyperbolic 3-manifolds

We focus now on the geometry which is of particular interest for us, the hyperbolic one. For a simpler exposition, from now on we say that a 3-manifold M without boundary is ***hyperbolic*** if it admits a complete hyperbolic structure of finite volume. It is worth recalling that Thurston in [TM79] proves that such a manifold, if it is not closed, is always the interior of a compact manifold $\bar{M}$ whose boundary consists of tori (or it is empty). Neighbourhoods of these boundary components are the *cusps* of M; if they are sufficiently small neighbourhoods, they can be taken to be *horoballs* in M, i.e. bounded by horospheres. When considering 3-manifolds with some torus boundary components, then, we will call them hyperbolic if their interior is.

By Mostow's Rigidity Theorem, there is at most one structure that makes a 3-manifold M hyperbolic. Therefore it makes sense to wonder about the *volume* of M, without specification of the hyperbolic structure (in order to avoid ambiguities due to rescaling, we will always consider the Riemannian metric of constant sectional curvature -1). We denote the volume by $\mathrm{vol}(M)$ (if M has boundary, with this notation we refer to the volume of its interior).

We now summarize the main facts about hyperbolic manifolds, mostly taken from [TM79].

1.2.1 Criteria for hyperbolicity

A 3-manifold M—for simplicity I do not include the possibility of M having boundary here—which admits a complete hyperbolic structure has the following properties (see also [Sco83]):

- it is irreducible (because its universal cover is $\mathbb{H}^3$: this implies that any 2-sphere in M, when lifted to the universal cover, bounds a 3-ball, which will project diffeomorphically to M);
- if $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then it corresponds to a toroidal end of M: in particular, M is atoroidal;
- M cannot be Seifert-fibered.

As a special case of Thurston's geometrization, the converse is also true:

Theorem 1.2.1. *Let M be a 3-manifold which is closed, or is the interior of a 3-manifold whose boundary consists of tori. Suppose M is irreducible, atoroidal, and not Seifert-fibered. Then M is hyperbolic.*

Thurston himself proved some important particular cases of this result:

Theorem 1.2.2 (Hyperbolization Theorem). *Let M be a 3-manifold which is closed, or is the interior of a 3-manifold whose boundary consists of tori. Suppose M is irreducible, atoroidal, and Haken. Then M is hyperbolic.*

1.2.2 Mapping tori

Thurston also dealt with another special case in [Thu98]: the one of ***mapping tori***, namely 3-manifolds M that can be topologically described as the quotient $S \times I / \sim_\psi$, where $\psi : S \to S$ is a suitable homeomorphism, or mapping class, and $\sim_\psi$ is the equivalence relation that identifies each point $(x, 0)$ with $(\psi(x), 1)$. A mapping torus is in other words a fiber space over S^1, with fiber S. The map ψ is called the ***monodromy*** of the mapping torus.

The JSJ decomposition and the related geometric properties for the manifold M will only depend on the surface S, and the homeomorphism ψ. We distinguish among three kinds of behaviour for ψ:

Definition 1.2.3. Let $\phi : S \to S$ be a homeomorphism of a surface. Suppose S is the interior of a compact surface $\bar{S}$, possibly with boundary.

- ϕ has *finite order* if some $n > 0$ exists such that ϕ^n is isotopic to id_S.
- ϕ is *reducible* if there is a finite set of homotopy classes of essential curves (i.e. homotopically non-trivial ones which are not homotopic into a boundary component of $\bar{S}$) which is fixed by ψ.
- ψ is *pseudo-Anosov* if neither it has finite order nor it is irreducible.

Actually there is an intrinsic definition for pseudo-Anosov homeomorphisms, but we will not use it. Each of the three possibilities above has a precise consequence on the JSJ characterization of a mapping torus:

Theorem 1.2.4. *Let M be a mapping torus, equal to $S \times I / \sim_\psi$ where S is a surface with $\chi(S) < 0$, and ψ is a self-homeomorphism of S. Then*

- *M is Seifert-fibered if and only if ψ has finite order;*
- *M contains an essential embedded torus if and only if ψ is reducible;*
- *M is hyperbolic if and only if ψ is pseudo-Anosov.*

The actual content of the theorem is the last point (the other two are easier).

1.2.3 Dehn surgery

Let M be a 3-manifold such that ∂M has a torus component T; and let $\alpha, \beta \in H_1(T)$ be a basis. Recall that a *Dehn surgery* (see [TM79]) on T with *slope* p/q with respect to the selected basis, where p, q are coprime integers (possibly p or $q = 0$), consists in the following construction: one plugs a solid torus D into T in such a way that the meridian curve of ∂D is glued to a curve of T which corresponds to $p\alpha + q\beta \in H_1(T)$. Conventionally, we identify the Dehn surgery with slope ∞ (to be considered as distinct from the one with slopes $\pm 1/0$) with no Dehn surgery at all.

In the following we summarize some fundamental facts about hyperbolic 3-manifolds and related techniques in particular the ones of [TM79].

Theorem 1.2.5 (Thurston). *Suppose M is a hyperbolic 3-manifold, with ∂M consists of a collection of tori. For each boundary torus there is a finite collection of slopes such that, if a Dehn surgery is performed on each boundary component with a slope different from the selected ones, then the resulting manifold is hyperbolic.*

In the setting of link complements in S^3 (more details below), if T is the boundary of a neighbourhood of a component of the link, then there is a preferred basis for $H_1(T)$; namely, $\alpha = m$ the meridian of the selected component, and $\beta = l$ its longitude.

1.2.4 Hyperbolic volume and Gromov norm

Definition 1.2.6. For a compact n-manifold M, let $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$ its fundamental class, and let $z = \sum_{j=1}^{k} a_j \sigma_j \in C_n(M, \partial M; \mathbb{R})$ a singular n-chain representing $[M, \partial M]$. We define $\|z\| := \sum_{j=1}^{k} |a_j|$; and we define the *Gromov norm* of M to be

$$\|[M, \partial M]\| := \inf \{ \|z\| \, | z \in C_n(M, \partial M; \mathbb{R}) \text{ represents } [M, \partial M] \} .$$

We recall the following results of Thurston (Proposition 6.5.2, Corollary 6.5.3 and Lemma 6.5.4 of [TM79]), that relate hyperbolic volume with Gromov norm and with Dehn surgery:

Proposition 1.2.7. *Let $\bar{M}$ be a possibly disconnected compact 3-manifold with boundary consisting of tori; and let $\bar{N}$ be a 3-manifold obtained from $\bar{M}$ by gluing some pairs of tori in $\partial \bar{M}$. Then*

$$\|[\bar{N}, \partial \bar{N}]\| \leq \|[\bar{M}, \partial \bar{M}]\|.$$

Actually Thurston's statement is about a different version of Gromov norm, but Gromov's Equivalence Theorem shows that it is equivalent to the regular one in this case.

Corollary 1.2.8. *For $\bar{M}$ a compact Seifert fibered 3-manifold, $\|[\bar{M}, \partial \bar{M}]\| = 0$.*

Theorem 1.2.9. *For M a hyperbolic 3-manifold which is the interior of a compact 3-manifold $\bar{M}$,*

$$\|[\bar{M}, \partial \bar{M}]\| = \frac{\mathrm{vol}(M)}{v_3}.$$

As a consequence, if a Dehn surgery on a boundary component of a hyperbolic manifold M results in a new hyperbolic manifold $\tilde{M}$, then $\mathrm{vol}(\tilde{M}) \leq \mathrm{vol}(M)$. If either, or both the two manifolds are not hyperbolic, it still holds that $\|[\tilde{M}, \partial \tilde{M}]\| \leq \|[M, \bar{M}]\|$. A much more precise statement is true, which involves *geometric convergence* of Riemannian 3-manifolds:

Theorem 1.2.10. *Let M be a 3-manifold with boundary, whose interior is hyperbolic. Then ∂M consists of tori; let C be one of these. We denote by $M(p,q)$ the interior of the 3-manifold obtained by Dehn surgery on C with coefficient (p,q). Then the following holds.*

- *If $M(p,q)$ admits a complete hyperbolic structure with finite volume, then* $\mathrm{vol}\,(M(p,q)) < \mathrm{vol}\,(\mathrm{int}(M))$.
- *There is a sequence of (p_i, q_i) such that the corresponding $M(p_i, q_i)$ geometrically converge to* $\mathrm{int}(M)$ *and in particular* $\mathrm{vol}\,(M(p_i, q_i)) \nearrow \mathrm{vol}\,(\mathrm{int}(M))$.

1.2.5 Most links are hyperbolic

We now restrict to the kind of manifolds we will be interested in the following, namely complements of knots and links in S^3. For a better concreteness, we define the *exterior* of a link K, which we will denote by $S^3 \setminus K$, as $S^3 - \mathcal{N}(K)$, where $\mathcal{N}(K)$ is an open tubular neighbourhood of K: so it is a compact manifold with torus boundary components. Clearly, the interior of a link exterior is homeomorphic to the mere link complement.

It is rather common that these manifolds are hyperbolic:

Theorem 1.2.11 (Thurston). *Every non-split link in S^3 satisfies one and only one of the following:*

- *it is a torus link;*
- *there is an essential torus embedded in its exterior;*
- *its complement admits a complete hyperbolic metric with finite volume: in this case we call it a hyperbolic link.*

In particular, this theorem implies that hyperbolic links are always prime, and that they are in some sense the 'widest' class of links.

1.3 Positive braids

Our focus will be in particular on positive braids, a kind of link diagrams that showed several reasons for interest.

Definition 1.3.1. A *positive braid* is a braid whose crossings are all positive ().

Such a braid diagram, say with n strings, corresponds to a word in a braid group

$$B_n = \langle \sigma_1, \ldots, \sigma_n \,|\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2;\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-1\rangle$$

only involving positive powers of the generators σ_i. In what follows we will, sometimes, commit an abuse of language by identifying a braid, or a positive braid, with its closure, which is a link in S^3 (or $D^2 \times S^1$). But the object we will be actually referring to will be clear.

1.3.1 Basic properties

Among the interesting properties of positive braids are the following:

Lemma 1.3.2. *Let b_1, b_2 be two positive words in the generators of B_n, corresponding to the same element of B_n. Then b_1, b_2 have the same length. In other words, two positive braids which are equivalent have the same number of crossings. Furthermore, if b_3 is any non-positive word in the generators of B_n, still corresponding to the same element of B_n as above, then it is longer than b_1.*

Proof. There is a unique well-defined group homomorphism $\psi : B_n \to \mathbb{Z}$ such that $\psi(\sigma_i) = 1$ for all the generators σ_i. Since b_1 and b_2 are positive words, their image through ϕ is their word length; and since b_3 is not positive, its image is less than its word length. But $b_1 = b_2 = b_3$ as elements of the group B_n, therefore $\phi(b_1) = \phi(b_2) = \phi(b_3)$. □

Theorem 1.3.3. *If L is a link in S^3 with a diagram D which is closure of a positive braid, then $S^3 \setminus K$ is fibered, with fiber the surface obtained with Seifert's algorithm from the diagram D.*

This is a consequence of the easier direction of Giroux's theorem, according to which all surfaces S in S^3 which are obtained one from another by a sequence of plumbings and deplumbings of Hopf bands are fibers for $S^3 \setminus \partial S$. A sequence of plumbings for a positive braid can be made explicit, and will result exactly in the surface obtained with Seifert's algorithm. In particular, such a surface is orientable.

The fiber structure of the complement of a positive braid is used in proving the following theorem by Cromwell (see [Cro93]), which is worth mention to communicate why positive braids are likely to give several information about the underlying link:

Theorem 1.3.4. *The closure of a positive braid is a diagram of a composite link if and only if it is* decomposable *i.e. if there exists a simple closed curve on the plane of the diagram which intersect it in exactly 2 points, and separates a non-trivial diagram of a tangle on both sides.*

1.3.2 Garside theory

The point of Garside theory is examining braids, in particular the positive ones, according to a canonical decomposition into simpler sub-braids (see [EM94]). Here are the 'bricks' of this decomposition:

Definition 1.3.5. A *permutation braid* is a braid such that each two strings cross each other at most once.

Positive permutation braids on n strings have a natural one-to-one correspondence with the symmetric group on n elements $\mathcal{S}_n$. Each elements $\pi \in \mathcal{S}_n$ can indeed be regarded as a picture where two horizontal rows of n points each are connected by segments, each of them joining the j-th point in the first row with the $\pi(j)$-th one in the second one. After having slightly deformed the segments in order to avoid that 3 of them meet at the same point, we resolve each crossing in a positive way and in so doing we obtain a positive permutation braid.

Among the positive permutation braids on n strings, we denote Δ_n the one such that *all* pairs of strings cross exactly once (therefore corresponding to the permutation $\pi(j) = n+1-j$). It is simple to subdivide a positive braid into positive permutation braids, but the result is far from being canonical: therefore we add the further request that the decomposition is *left-weighted*, namely: for any two consecutive factors A_iA_{i+1} in the decomposition, if A_{i+1} can be written with a word that begins by σ_j, then A_i can be written with a word that ends by the same σ_j.

Theorem 1.3.6. *Each positive braid p admits a unique decomposition $p = \Delta^k a_1 \dots a_l$ where the a_j are non-trivial positive permutation braids not equal to Δ, and $a_1 \dots a_l$ is a left-weighted decomposition.*

This statement can be adapted to get a decomposition for *any* braid: for a braid B, it is always possible to find a least positive integer m such that $b\Delta^m$ is a positive braid. If the latter admits a left-weighted decomposition $\Delta^k a_1 \dots a_l$, then $b = \Delta^{k-m} a_1 \dots a_l$ (by Δ^{-1} we mean a negative permutation braid built similarly as Δ, but using negative crossings).

Chapter 2

Hyperbolic volumes and pants complexes

2.1 The curve complex and its relatives

A good way of studying a 3-manifold is, quite often, focusing on some surface embedded in it. In particular it might be worth considering some simplicial complexes associated to such a surface. Their definitions might be colliding from an author to another; we follow the ones in [MM00]. We will use here the common point of view of considering only *orientable* surfaces without boundary, but regarding them as the interior of compact surfaces with boundary (in particular, by *puncture* we mean a collar neighbourhood of a boundary component). The definitions of those simplicial complexes are usually different according to the *complexity* of the surface, namely the quantity $\xi(S) := 3\gamma(S) + p(S)$, where γ denotes the genus, and p the number of punctures.

2.1.1 The curve complex and the subsurface projection

Definition 2.1.1. Let S be an orientable surface, different from the torus. The *curve complex* $\mathbf{C}(S)$ of S is a simplicial complex whose vertices correspond to isotopy classes of essential, simple closed curves in S.

- If $\xi(S) > 4$, there is an edge between two vertices if and only if the corresponding curves can be isotoped to be disjoint.
- If S is a torus with 1 puncture, there is an edge between two vertices if and only if the corresponding curves, when isotoped into minimal position, intersect in 1 point.
- If S is a sphere with 4 punctures, there is an edge between two vertices if

and only if the corresponding curves, when isotoped into minimal position, intersect in 2 points.

- If $\xi(S) = 3$, the curve complex is empty.

The higher dimensional skeletons of $\mathbf{C}(S)$ are defined inductively by saying that $n+1$ vertices are the vertices of a n-simplex in the n-skeleton if and only if the $n-1$-skeleton includes all the faces of the simplex.

A *subdomain* Y of a surface S with $\xi(S) \geq 4$ is a connected, open subsurface, which is either the whole S, or is such that its boundary in S consists of essential curves (including annuli, which are bounded by 2 curves which are isotopic to each other). There is a projection $\pi_Y : \mathbf{C}(S) \to \mathcal{P}(\mathbf{C}(Y))$; if Y is not an annulus, this projection is such that $\pi_Y(\alpha) = \alpha$ if α is a curve which is essentially contained in Y; $\pi_Y(\alpha) = \varnothing$ if α is essentially disjoint from Y; and if α intersect ∂Y essentially, then $\pi_Y(\alpha) = \partial\mathcal{N}(\alpha \cup \partial Y) \cap Y$.

A special definition of curve complex is given for an annulus Y which is subdomain of some S. Consider the annular cover $\tilde{Y}$ of S to which Y lifts homeomorphically. Pick a hyperbolic metric on S: this will allow definition of a closure $\hat{Y}$ of $\tilde{Y}$, defined as the quotient of the closure of the universal cover $\mathbb{H}^2$ of S. Now, vertices of $\mathbf{C}(Y)$ will correspond to arcs properly embedded into $\hat{Y}$, up to isotopies that keep their extremes fixed. Again, there is an edge between two vertices each time the corresponding arcs can be isotoped to be made disjoint, and the same construction as above applies for higher dimensions. If $\alpha \subset S$ is a curve, we will use the notation $\mathbf{C}(\alpha) := \mathbf{C}(\mathcal{N}(\alpha))$. In this case $\pi_Y(\alpha)$ is empty if α does not intersect essentially the core of Y; otherwise it is defined as the complete lift of the curve α to $\hat{Y}$.

2.1.2 Marking and pants graphs

We can now talk about another kind of graph based on curves of an oriented surface. A *pants decomposition* for an oriented surface S with $\xi(S) \geq 4$ is a maximal collection of essential, pairwise disjoint isotopy classes of curves $\{\alpha_1, \ldots, \alpha_n\}$ in S. A *complete marking* is a collection of pairs $\{p_1 = (\alpha_1, t_1), \ldots, p_n = (\alpha_n, t_n)\}$ such that $\{\alpha_1, \ldots, \alpha_n\}$ is a pants decomposition of S and, for each i, $t_i \subset \mathbf{C}(\alpha_i)$ (called *transversal*) is a nonempty set of diameter 1. A complete marking is *clean* if, for each i, a curve $\beta_i \subset S$ exists such that: $\mathcal{N}(\alpha_i \cup \beta_i)$ is either a 1-punctured torus or a 4-punctured sphere; $\pi_{\alpha_i}(\beta_i) = t_i$; and β_i is disjoint from α_j for all $j \neq i$. If such β_i's exists, they are unique.

If a complete marking is not clean, a clean marking which is *compatible* with it is one that has the same base pants decomposition, and each transversal set has the minimum distance possible from the original one, among all the clean complete markings with the same base pants decomposition. For a subdomain $Y \subset S$, the projection $\pi_Y(\mu)$ of a marking μ is defined as the union of the projections of the curves in the underlying pants decomposition, when Y is not an annulus. Otherwise, when Y is a regular neighbourhood of some α_i appearing in the pants decomposition underlying M, then $\pi_Y(\mu) = t_i$. When it is not, $\pi_Y(\mu) = \varnothing$. A simplified version of this definition is given for pants decomposition.

Definition 2.1.2. The *marking graph* $\mathbf{M}(S)$ of an orientable surface S with $\xi(S) \geq 4$ is a graph whose vertices correspond to all possible clean complete markings on S. There is a vertex between each couple of markings $((\alpha_1, \pi_{\alpha_1}(\beta_1)), \ldots, (\alpha_n, \pi_{\alpha_n}(\beta_n)))$ and $\left((\alpha_1', \pi_{\alpha_1'}(\beta_1')), \ldots, (\alpha_n', \pi_{\alpha_n'}(\beta_n'))\right)$ that are obtained from each other with one of these moves:

- twist: the only difference between the two markings is that β_i' is obtained from β_i by performing a twist or a half twist (depending on $\#\alpha_i \cap \beta_i$) around α_i;
- flip: all the couples are the same, except for one where the roles of α_i and β_i have been switched, and the new complete marking has been then replaced with a compatible clean one.

The *pants graph* $\mathbf{P}(S)$ of S is a graph whose vertices correspond to all possible pants decompositions S. Two vertices are joined by an edge if and only if the corresponding pants decompositions can be completed to complete markings which are obtained from each other with a flip move.

Both graphs (or, to better say, their 0-skeletons) are made into metric spaces by considering the natural graph distance. For Y a subdomain of S, we define an improper notion of distance between projections: $d_Y(A, B) \coloneqq \operatorname{diam}_{\mathbf{C}(Y)}(\pi_Y(A), \pi_Y(B))$ where A, B are elements or subsets of $\mathbf{C}(S), \mathbf{M}(S)$ or $\mathbf{P}(S)$.

Brock in his paper [Bro03] points out that $\mathbf{P}(S)$ is quasi-isometric to the Teichmüller space of S with the Weil-Peterson metric. For p a pants decomposition of S and $L > 0$, let $V_L(p)$ be the open subset of $\operatorname{Teich}(S)$ consisting of all structures such that every curve in p is isotopic to a geodesic of length $< L$. According to a result of Bers, there is some L_0 such that the $V_{L_0}(p)$, varying p, cover the whole $\operatorname{Teich}(S)$.

Theorem 2.1.3. *Consider a map*

$$Q : \mathbf{P}_0(S) \to \mathrm{Teich}(S)$$

where $\mathbf{P}_0(S)$ *is vertex set of the pants graph, with its natural graph metric; consider the Weil-Peterson metric on* Teich(S). *Suppose that, for each* p, $Q(p) \in V_{L_0}(p)$*; then* Q *is a quasi-isometry.*

Masur and Minsky in [MM00] prove that both distances in marking and pants graphs are expressible in terms of subsurface projection:

Theorem 2.1.4. *Let* S *be an orientable surface with* $\xi(S) \geq 4$. *There exists a constant* $C_0(S)$ *such that, given* $C > C_0$, *there are constants* e_0, e_1 *only depending on it such that, for any pair of complete clean markings* μ_I, μ_T:

$$e_0^{-1} d_{\mathbf{M}}(\mu_I, \mu_T) - e_1 \leq \sum_{Y \subseteq S:\ d_Y(\mu_I, \mu_T) > C} d_Y(\mu_I, \mu_T) \leq e_0 d_{\mathbf{M}}(\mu_I, \mu_T) + e_1;$$

and, for any pair of pants decompositions p_I, p_T,

$$e_0^{-1} d_{\mathbf{P}}(p_I, p_T) - e_1 \leq \sum_{\substack{Y \subseteq S:\ d_Y(p_I, p_T) > C \\ Y \text{ is not an annulus}}} d_Y(p_I, p_T) \leq e_0 d_{\mathbf{P}}(p_I, p_T) + e_1.$$

2.2 Brock's estimate on hyperbolic volumes

Brock's original result about hyperbolic volume (in [Bro03]) deals with quasi-Fuchsian hyperbolic 3-manifolds, but in [Bro01] he gives an easy way of converting it into a result about mapping tori, therefore fit to be exploited in the context of complements of positive braids. Before mentioning the results of Brock, it might be worth recalling some concepts: a ***quasi-Fuchsian group*** is a Kleinian group $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ whose limit set is a topological circumference $\gamma \subseteq \partial\mathbb{H}^3$. The latter is identified with $\mathbb{CP}^1$. It is easily shown that, if A, B are the two components of $\mathbb{CP}^1 - \gamma$, then A/Γ and B/Γ are homeomorphic Riemann surfaces, with a canonical identification of their fundamental groups with Γ. By Bers' simultaneous uniformization theorem ([Ber60]), every time one picks an oriented surface S with $\chi(S) < 0$ and two elements $X, Y \in \mathrm{Teich}(S)$, there is a unique conjugacy class of quasi-Fuchian subgroup Γ in $\mathrm{Isom}^+(\mathbb{H}^3)$ that realizes X, Y as the two boundary Riemann surfaces, and induces the right identification of their fundamental groups. The hyperbolic 3-manifold obtained as the quotient of $\mathbb{H}^3$ by Γ is denoted by $Q(X, Y)$ and is naturally homeomorphic to $S \times I$. Its *core*, denoted by $C(X, Y)$, is the convex hull of the limit set of Γ; it is again homotopically equivalent to S.

Brock proves the following:

Theorem 2.2.1 (Brock's theorem for hyperbolic cores). *Let S be an oriented surface, and pick $X, Y \in \operatorname{Teich}(S)$. Let p_X, p_Y be pants decompositions of S such that $X \in V_{L_0}(p_X)$, $Y \in V_{L_0}(p_Y)$ (using the notation of last section). Then two constants $c_0, c_1 > 0$, only depending on S, exist such that*

$$c_0^{-1} d_{\mathbf{P}}(p_X, p_Y) - c_1 < \operatorname{vol}(C(X, Y)) < c_0 d_{\mathbf{P}}(p_X, p_Y) + c_1.$$

Theorem 2.2.2 (Brock's theorem for mapping tori). *Let S be an oriented surface with $\chi(S) < 0$, and let $M = S \times I / \sim_\psi$ where $\psi : S \to S$ is a pseudo-Anosov self-homeomorphism. Then two constants $k_0, k_1 > 0$, only depending on S, exist such that*

$$k_0^{-1} \|\psi\| - k_1 < \operatorname{vol}(M) < k_0 \|\psi\| + k_1.$$

Here, $\|\psi\|$ denotes the translation distance of the map induced by ψ on either $\operatorname{Teich}(S)$ *(with the Weil-Peterson metric) or* $\mathbf{P}(S)$ *(with the graph metric).*

We now summarize very briefly what are the main ideas in the proof of the first theorem.

Lower bound. The core of the proof is the following lemma:

Lemma 2.2.3. *Let $g = (p_I = p_0, p_1, \ldots, p_N = p_T)$ be a sequence of pants decompositions of S such that the distance between any two consecutive ones is less than a fixed k. Let $\mathcal{S}_g$ be the set of all isotopy classes of curves in S which appear in some of the pants decompositions in g. Then $d_{\mathbf{P}}(P_I, P_T) < K_0 |\mathcal{S}_g|$.*

In other words, the distance in the pants complex will have an upper bound not depending on the length of the sequence N, but only on the total number of different curves which are involved. The proof of this lemma is rather technical and leans on the statement of Theorem 2.1.4 for pants graphs.

A way of obtaining such a sequence is the following. The hyperbolic structures X and Y can be realized, with singularities, within $C(X, Y)$; and they can be connected by a path of singular hyperbolic surfaces. This corresponds to a path of 'regularized' hyperbolic surfaces in $\operatorname{Teich}(S)$, which is covered by a finite family of $V_{L_0}(p_j)$'s (see above). The corresponding sequence of p_j allows application of the lemma. But the length of the curves in the collection $\mathcal{S}_g$ will be (almost by definition) at most L_0; and it is easily shown that (apart from additive and multiplicative constants) the amount of such curves in $C(X,Y)$ is bounded by its volume. $\square$

Upper bound. The upper bound on the volume is obtained with a subdivision of $C(X,Y)$ into tetrahedra. More precisely, Brock fixes a geodesic in the pants graph $\{p_0 = p_X, p_1, \ldots, p_n = p_Y\}$ (where, of course, $n = d_{\mathbf{P}}(p_X, p_Y)$) and considers singular hyperbolic surfaces X_j in $C(X,Y)$, each one realizing the curves in p_j as proper geodesic curves in $C(X,Y)$. These surfaces can be seen as the surfaces $S \times \{j/n\} \subset S \times I \cong C(X,Y)$. Then, since each p_j differs from the following p_{j+1} by an elementary move, there is a standard way of building a triangulation between two consecutive 'layers'. After straightening this triangulation, it is shown all tetrahedra can be supposed to have very small volume, except for a fixed number of them for each layer. So we have a constant bound to how much can each layer contribute to the total volume of $C(X,Y)$; and the layers are as many the vertices of the selected geodesic in the pants complex. $\square$

The proof is basically the same for the theorem about mapping tori; one must select a singular hyperbolic copy of S in M and then, apart for some slight adaptations, the proof goes through as above, connecting a pants decomposition p to its translated $\psi(p)$.

2.3 Braids in solid tori

2.3.1 How to use Brock's result

Even if Brock's result applies for positive braids in particular, because they are fibered, it turns out to be difficult to find a bound on the translation distance induced by the monodromy map ψ, in terms of diagrammatic quantities. Moreover Brock's constants depend on the topology of the fiber, and there is no simple control for it either.

Another approach, perhaps more suitable to this kind of arguments, might be the following instead: we start with a link, depicted as the closure of a braid B (usually a positive one for our aims, but not necessarily), and consider a curve C encircling all the strings of B. Let $\hat{B} \coloneqq B \cup C$: then $S^3 \setminus \hat{B}$—which can also be thought as the exterior of B embedded in a solid torus—is fibered over a punctured closed disc

$\bar{S}$ (whose interior we denote S) i.e. the disc bounded by a curve parallel to C and transversely intersected by the braid strings.

A reasonable approach to our problem is then to find out when $S^3 \setminus \hat{B}$ is hyperbolic, and to give bounds on its volume according to Brock's result. The number of punctures is the number n of strings of the considered braid. Therefore, once this has been fixed (for technical reasons we take it to be ≥ 3) the volume of the complement of a braid in the solid torus—provided it is hyperbolic—is, up to constants, the translation distance $\|\psi\|_{\mathbf{P}}$ induced by the monodromy $\psi : S \to S$ in the pants complex.

The original positive braid complement $S^3 \setminus B$ can then be obtained via Dehn surgery along C. Under mild further assumptions, $S^3 \setminus \hat{B}$ and $S^3 \setminus B$ are not so different (see [BH96] and [FKP08]):

Theorem 2.3.1 (Gromov-Thurston's 2π Theorem). *Let M be a complete hyperbolic 3-manifold of finite volume, and $P_1, \ldots, P_v$ disjoint horoball neighbourhoods of the cusps of M. Suppose r_i is a slope on the torus ∂P_i represented by a geodesic α_i with length $> 2\pi$ in the Euclidean metric, for each $i = 1, \ldots, v$. Then the 3-manifold $M(r_1, \ldots, r_v)$ obtained by Dehn surgery on the selected cusps P_i with the specified slopes r_i admits a complete hyperbolic metric.*

According to what happens in general when doing Dehn surgery, in this case $\mathrm{vol}\,(M(r_1, \ldots, r_v)) < \mathrm{vol}(M)$. But there is also a bound in the opposite direction:

Theorem 2.3.2 (Futer, Kalfagianni, Purcell). *In the situation depicted in the previous theorem, let $\ell_{\min} > 2\pi$ be the geodesic length of the shortest slope. Then*

$$\mathrm{vol}\,(M(r_1, \ldots, r_v)) \geq \left(1 - \left(\frac{2\pi}{\ell_{\min}}\right)^2\right)^{3/2} \mathrm{vol}(M).$$

The reason why one can hope that the geodesic length of the curves involved is large enough is the following theorem by Futer and Schleimer ([FS11]).

Theorem 2.3.3. *Let M be a mapping torus with fiber S and pseudo-Anosov monodromy ϕ (therefore M has a hyperbolic structure). Suppose S has a puncture p that is fixed by $\phi^n =: \psi$. Let P be the maximal horoball neighbourhood corresponding to p. Then all geodesic closed curves on ∂P, except its longitude (i.e. the only one that is homotopic into a fiber) are longer than $\|\psi\|_{\mathbf{A}} / \left(536\chi(S)^4\right)$.*

In the statement, $\|\psi\|_{\mathbf{A}}$ is the translation distance of ψ in the *arc complex* $\mathbf{A}(S)$, which is defined similarly as the curve complex, except that the vertices in its 0-skeleton are taken to be homotopy classes of properly embedded arcs in S having both endpoints in some puncture. In our context (we take, of course, p to be the puncture corresponding to the 'exterior' of the fiber punctured disc S, and $\psi = \phi$), it is likely that $\|\psi\|_{\mathbf{A}}$ is high for positive braids which are complicated enough—in some suitable sense.

2.3.2 Monodromy from the braid diagram: naïve attempts

Now, of course, the goal would be relating ψ with the positive braid diagram: in particular we would like to have a way to use such a diagram for telling whether it is pseudo-Anosov; and, if so, for estimating $\|\psi\|_{\mathbf{P}}$. In this whole subsection, S will denote the punctured open disc transversal to the considered braid, as defined above.

There is a canonical isomorphism (see [FM11]) between the braid group B_n and the mapping class group $\mathrm{Mod}(\bar{S})$ of the closure of that disc. The isomorphism is given by the map that sends each σ_j to the counter-clockwise half-twist of $\bar{S}$ around the j-th and the $(j+1)$-th punctures. A similarly defined map establishes an isomorphism $\mathrm{Mod}(S) \cong {}^{B_n}\!/_{Z(B_n)}$, where the center of B_n is the subgroup generated by $(\sigma_1 \ldots \sigma_{n-1})^n$. This braid, indeed, induces a Dehn twist around the boundary of $\bar{S}$.

Remark 2.3.4. We can think of this homomorphism as associating to each braid the homeomorphism of $\bar{S}$ obtained by pushing $\bar{S}$ from the bottom of the braid to the bottom. It is necessary that we take this point of view rather than pushing $\bar{S}$ the other way around, otherwise we would have an *anti*homomorphism. $\diamondsuit$

Saying that ψ is pseudo-Anosov is equivalent to saying that the braid is *irreducible*. A braid is reducible if there are an essential curve γ in the copy S_0 of S corresponding to the top of the braid, and a number $m > 0$, for which the following holds: push the curve down along the braid, and then up to reach S_0 again; after doing this m times, we get a curve which is homotopic to γ. Such a curve is called a *reducing curve*. A reducing curve is *essential* if any other reducing curve can be isotoped off it; it can be shown that, if a braid is reducible, there exists some essential reducing curve for it.

A paper by González-Meneses and Wiest ([GMW10]) gives a couple of results concerning control of such curves:

Theorem 2.3.5. *Let $b \in B_n$ be a non-periodic, reducible braid. Then a computable subset $SC^{[m]}(x)$ of the conjugacy class of b in B_n will have the property that every element $b' \in SC^{[m]}(x)$ admits an essential reduction curve which is either round or almost round.*

Here, we suppose to have fixed once and for all an identification of the fiber S with the unit disc in $\mathbb{C}$, in such a way that all its punctures lie on $\mathbb{R}$, in the same order as the corresponding braid strings. A curve γ in S (or a copy of it) is called ***round*** if it is homotopic to a geometric circumference in S; it is ***almost round*** if it could be obtained by pushing a round curve along a positive or negative permutation braid.

And, speaking about positive braids only, we have the following result:

Proposition 2.3.6. *If γ is an essential reduction curve for a positive braid b, and the strings of b which are encircled by γ do not cross each other, then γ is either round or almost round.*

The proof of this proposition, in particular, considers arcs in $S_0 \cup$ punctures that cross horizontally exactly the punctures encircled by γ. The hypothesis on b implies that such an arc, when slid along γ, eventually returns to its initial position. But this is far from being true for positive braids in general. Also, such a strong claim is surely false for positive braids in general.

Remark 2.3.7. Positivity of a braid does not give any restriction on the homeomorphism it induces on S. Any homeomorphism of S can be identifed with a word in B_n but, since the word $(\sigma_1 \dots \sigma_{n-1})^n$ is trivial as an element of $\mathrm{Mod}(S)$, one can replace each negative letter in that word with a positive subword, and in so doing obtain a positive braid that induces the desired homeomorphism. The difference from the original word can only be seen if we consider the homeomorphism induced on $\bar{S}$: the new word will differ from the original one by some Dehn twists parallel to the boundary. Since in the context we are eventually interested in, S has no boundary, the only benefit we might hope to obtain from positive braids is that they are a kind of diagram 'rigid' enough to carry actual information about the induced homeomorphism. $\Diamond$

In particular, given any reducible positive braid b, select a reduction curve for it, and let ϕ be a self-homeomorphism of S that makes it become complicated at will. Suppose ϕ is induced by a positive braid b_1, and ϕ^{-1} is induced by a positive braid

b_2. Then $b_2 b b_1$ has a reduction curve that is complicated at will. For this reason, I have been working on a weaker conjecture for recognizing reducible braids:

Conjecture 2.3.8. Suppose that positive braid b is reducible, and let γ be an essential reducing curve. Then there is an initial subword b' of b such that γ, slid along b', is turned into a round curve.

A necessary thing to say is that this conjecture dates back to the very beginning of my studies on positive braids; therefore it has been stated when I was still unaware of most of the results in this field I have been stating above. Actually, at the beginning, we believed positivity of a braid to be a stricter condition that what it actually is. In particular, we thought that a positive braid could not fix any curve at all, unless there was some really easily recognizable 'signal' in its diagram, or expression as a word, that indicated the opposite. Even more than that, we thought that it was easy to find some hypothesis on positive braids such that the induced homeomorphism had a high translation distance on $\mathbf{C}(S)$. Unfortunately, this turned out not to be the case. But this conjecture, or some versions of it, *might* still hold, because in hindsight it is a way of saying that, even if Proposition 2.3.6 cannot hold for any positive braid, the trouble with it is just that a positive braid for which the claim is true can be composed with a pre- and a post-braid that mess up the reduction curves.

Another motivation for this conjecture is some empirical observation. When sliding a curve in S down along a positive braid, it tends to become more and more tangled, essentially because it is subject to a sequence of half-twists that are all performed in the same sense. Of course this does not mean that the curve cannot be untangled in order to get back to its original position; but that the untangling is far from being straightforward, and some experiments lead to the supposition that there is no way of modifying a curve, and eventually resetting it to the original configuration, if the procedure does not make the curve into round at some stage.

I have been trying to solve this conjecture using a couple of different formalisms, but neither of them looks to be powerful enough to give an idea of the constraints induced by positive braids on reduction curves. The trouble is also that for a long time I have been trying to prove that the claim holds for an *outermost* reducing curve, rather than an essential one. This is, in some way, a more immediate condition to confront with, but farther from the results of González-Meneses.

The first formalism is a combinatorial model of curves on S, similar in spirit to normal surfaces. If the considered braids have n strings, we identify S with the

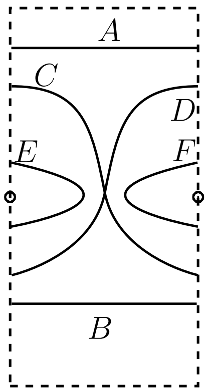


Figure 2.1: The six possible types of arcs in a rectangle R_j.

rectangle $\{x+iy|0<x<n+1,\ -1<y<1\} \subset \mathbb{C}$ from which we remove the integer points $1,\dots,n$, identified with the punctures. For $j=0,\dots,n$, let $R_j = \{x+iy|j< x<j+1,\ -1<y<1\} \subset R$. If we consider any collection of disjoint embedded curves in S, their intersection with each R_j will be a set of arcs, of 6 different types.

So, this collection is completely determined by a vector $(a_j,\dots,f_j)_{i\leq j\leq n} \in \mathbb{R}^{6n}$ specifying how many arcs of each type appear in each rectangle. Actually, it is simple to verify that a_j's and b_j's can be neglected because, given the other coefficient, there is not more than one consistent choice of the a_j's and b_j's. Furthermore the e_j's and f_j's can be replaced by $e'_j := e_j - f_j$, as at least one of these coefficient is 0 for each j. So a collection of curves in S is described by a vector in $\mathbb{R}^{3n}$. It is easy to understand when such a vector corresponds to a round curve, and there is another interesting property: consider two collections of curves Γ_1, and Γ_2, represented by vectors v_1, v_2. Then v_1+v_2 corresponds to the union of the two collection if no two curves intersect, otherwise v_1+v_2 either does not correspond to any collection of curves, or there is a curve from one of the two collections that does not appear in the collection represented by v_1+v_2.

The action of each braid generator σ_j on a curve collection represented by a vector v reflects into a function $v \mapsto A_j(v)$ which is piecewise linear, meaning that there exists a finite family of hyperplanes in $\mathbb{R}^{3n}$ such that, in the closure of each connected component of the complement, A_j is given by a matrix. But a composition of such A_j's does not look to have any 'manageable' properties.

The other approach I have been trying is adapting the approach used by Menasco

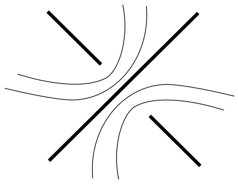


Figure 2.2: Menasco's curves near a crossing of a link diagram. The arcs must be in equal number on the two sides of the above segment of the diagram, because they are components of the intersection of the splitting/reducing surface with the half-sphere of T relative to the crossing.

in [Men84] for proving that alternating links are split, or prime, if and only if they appear to be so. This is an analogous result as what Theorem 1.3.4 by Cromwell asserts for positive braids.

Menasco's proof is based on representing a sphere splitting a link as a set of closed curves in the diagram plane that do not intersect the segments of the diagram. Such curves are obtained as follows: let T be the surface which coincides with the projection plane of the diagram away from its crossing, whereas in a neighbourhood of each of them the projection plane is replaced by a half-sphere in the positive half-space. So, we may regard the upper arcs of the crossings to lie on these half-spheres; and the lower ones to lie in the other half of the sphere, under the projection plane.

After an isotopy which ensures that the sphere is in general position with T, the collection of curves is simply obtained as the intersection of the two of them. A similar collection can be obtained looking at the other face of the diagram, i.e. considering a surface T' built similarly to T but taking the half-spheres in the negative half-space.

Menasco's approach is convenient for alternating diagrams: if one considers the sequence of half-spheres (i.e. of crossing neighbourhoods) a curve enters, the respective crossings will lie alternatively on its left and on its right. In the case of positive braids, a possible approach could be representing an essential embedded torus in the link complement, using the same kind of curves; and then, if the diagram is too complicated, finding out how can it be simplified to get a simpler torus. It is worth noting that Cromwell in [Cro93] applies a similar idea, but rather than analysing the curves of intersection with the diagram plane, he focuses on the ones induced on the fiber surface on the positive braid.

Back to our case, the diagram being a positive braid still imposes constraints on how the curves behave with respect to the crossings. In particular there are constraints on the behaviour of the curve with respect to the columns of the diagram i.e. the 'vertical strips' between its strands. When a curve enters a crossing neighbourhood, it is switching from a complementary region of the braid to one lying in an adjacent column: the one on the left if the curve goes through the crossing neighbourhood keeping *below* the SW-NE segment of the crossing; the one of the right if the curve keeps *above* that segment. But still, the curves in the diagram have an overly complicated appearance even in the simplest cases.

With hindsight, I realize that the biggest mistake in my approach was focusing on braids as words in the generators σ_i. This, in particular when applying the first of my two approaches, alters the geometric intuition of the action of braids on curves. At that time I was not even aware of their existence, but two better decomposition of a (positive) braid are the Garside's one into permutation braids, and the similar one of Dynnikov and Wiest ([DW04]), that I am going to introduce afterwards. Both decompositions allow a more natural interpretation of what happens when a curve is slid along a braid.

2.3.3 A remark: what if we switch to the marking graph?

Marking graphs of surfaces have been studied and exploited in the recent literature much more than pants graphs. For this reason, it might be worth having a way of understanding the translation distance in the pants graph from the one in the marking graph. A rough way of understanding the difference between the two is comparing the two statements of Theorem 2.1.4. Namely, if $p \in \mathbf{P}_0(S)$ is a pants decomposition, and $\mu \in \mathbf{M}_0(S)$ is a complete clean marking that projects to p, then roughly $d_{\mathbf{P}}(p, \psi(p))$ and $d_{\mathbf{M}}(\mu, \psi(\mu))$ are both expressed as a summation of distances of the subsurface projections: the difference between these two summations is

$$\sum_{\substack{Y \subseteq S : d_Y(\mu, \psi(\mu)) > C \\ Y \text{ annulus}}} d_Y(\mu, \psi(\mu)).$$

Unfortunately, the constants involved in the two statements do not allow us to see it as a rough measure of $d_{\mathbf{M}}(\mu, \psi(\mu)) - d_{\mathbf{P}}(p, \psi(p))$.

A nice property of the marking graph, that the pants graph does not have, is that the mapping class group of the surface acts on it in a co-compact and properly discontinuous way. Therefore, by an easy application of the Švarc-Milnor lemma, we have

Proposition 2.3.9. *Given a surface T there exist constants $c_0, c_1 > 0$ such that, given any finite generating set for* $\mathrm{Mod}(T)$ *and any $\mu_0 \in \mathbf{M}(T)$, for each $\psi \in \mathrm{Mod}(T)$ it holds that*

$$c_0^{-1} d_{\mathbf{M}}(\mu_0, \psi(\mu_0)) - c_1 < |\psi| < c_0 d_{\mathbf{M}}(\mu_0, \psi(\mu_0)) + c_1$$

where $|\psi|$ is the minimal word length of ψ with respect to the selected set of generators of $\mathrm{Mod}(T)$.

In particular, taking T to be our punctured disk S, the marking graph distance is given by the word length of the braid after projection to ${B_n}\big/{\langle(\sigma_1 \dots \sigma_{n-1})^n\rangle}$. With some *ad hoc* definition of the marking graph for $\bar{S}$, which is not difficult to find, we will have that the distance there is given by the word length in B_n.

2.4 Investigating complexity with train tracks

Another idea that looks worthwhile for getting estimates on the pants graphs of our interest is considering how do *train tracks* get modified by the action of the braid. This is inspired by the work of Dynnikov and Wiest on braid complexity [DW04] and by the one of Masur, Mosher and Schleimer [MMS12] which shows that a sequence of train track modifications which satisfies certain properties can be interpreted as a quasi-geodesic in the marking graph. The result can hopefully be adapted to a pants graph context.

2.4.1 Train tracks and quasi-geodesic paths

This subsection is a summary of what we use from [MMS12].

Definition 2.4.1. A *pretrack* τ on a compact surface S is a 3-valent topological graph, properly embedded in S in such a way that

- each edge (which we will call *branch*) is embedded smoothly;
- for each vertex (called *switch*) x of the graph, there is a $v_x \in T_x S$ such that all edges having x as endpoint have tangent direction there equal to $\pm v_x$;
- for each branch of τ there is a *train-route*, namely a smooth immersion $\mathbb{R} \to S$ with image contained in τ, passing through that branch.

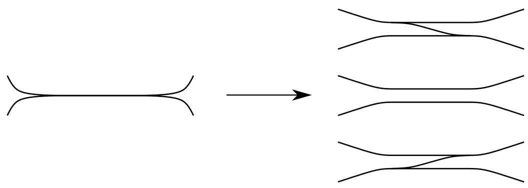
For T a connected subsurface of S bounded by piecewise smooth curves, we define its *index* to be $\mathrm{index}(T) = \chi(T) + (c^-(T) - c^+(T))/4$, where $c^-(T)$ and $c^+(T)$ are the number of inward and outward corners in the boundary of T, respectively.

We call a pretrack τ a *train track* if it is compact, every complementary region of τ in S has at least 1 corner and has negative index.

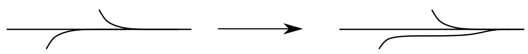
The condition on the complementary regions is only needed in order to avoid too simple ones. Now, according to the definition, in each vertex 3 ends of branches are joined: two of them will have the same tangent vector and will be called *small*, the remaining one will be called *large*. A branch whose ends are both large (small) will be called large (small) as well. Branches which are neither large nor small are called *mixed*.

We now describe the elementary operations that can take place on a train track (or a pretrack). Of course we will always consider them up to isotopy. Pictures come from [MMS12].

- If a branch is large, it can undergo a *split*, in 3 different ways:



- If a branch is mixed, it can undergo a *slide*:



- The inverse move of a split is called a *fold*.

Given a pretrack τ, a regular neighbourhood $\mathcal{N}(\tau)$, consisting of the union of a rectangle for each branch, can be considered. Each rectangle is naturally laminated into *ties* i.e. 'vertical' segments transversal to the branch. A pretrack σ is said to be *carried* by τ if σ can be isotoped into $\mathcal{N}(\tau)$ in such a way that it gets transversal to the ties. Similarly we may talk about a curve, or a multicurve, carried by τ.

A *transverse measure* on a train track τ is a function $w : \{\text{branches of } \tau\} \to \mathbb{R}_{\geq 0}$ such that, for each switch, if a is the branch giving its large end, and b, c are the ones giving its small ends, then $w(a) = w(b) + w(c)$. A multicurve carried by τ induces an integer transverse measure which assigns, to each branch of τ, the number of intersections of the multicurve with a tie transversal to it. It may be checked that the set of all transveral measures on a fixed train track is a cone in $\mathbb{R}^m$ (m being the number of branches of τ) over a convex polyhedron; and the lines corresponding to the vertices of this polyhedron each contain one minimal integer measure, induced by a curve carried by τ. Such a curve is called a *vertex curve*; we will denote the set of all vertex curves of τ by $V(\tau)$.

Masur, Mosher and Schleimer define the *train track graph* $\mathcal{T}(S)$. Its vertices are train tracks which are *birecurrent* (see the original paper for the definition) whose collection of vertex curves *fills* S i.e. cut it into a collection of topological discs and 1-punctured discs. Edges are pairs of train tracks obtained from each other with one elementary operation. The train track graph is a metric space in the usual way.

After that, they define a modified version of the marking graph, which they denote $\mathcal{M}(S)$: their construction is performed in such a way that

- the vertices of this graph are collections of curves on S that have small mutual intersection number and fill it; these include both the ones corresponding to complete clean markings, and the ones which are vertex curves of some train track; in particular there is an obvious map $\mathcal{T}(S) \to \mathcal{M}(S)$ taking $\tau \mapsto V(\tau)$;
- two vertices are joined by an edge if and only if they have a small intersection number with each other; the threshold is chosen in such a way that we will have an edge both if the two vertices are markings differing by one elementary move, and if they correspond to train track differing by one elementary move;
- there is a natural quasi-isometry $\mathcal{M}(S) \to \mathbf{M}(S)$.

We can finally summarize (and slightly restate) the theorems in [MMS12] of our interest:

Theorem 2.4.2. *For any surface S with $\xi(S) \geq 1$ there are a constant Q and a function $f : \mathbb{N} \to \mathbb{R}_+$ such that the following holds. If $\{\tau_i\}_i$ is a sliding and splitting sequence in $\mathcal{T}(S)$ such that there are no more than a consecutive splittings, then, setting $Q' = Q + f(a)$, we have*

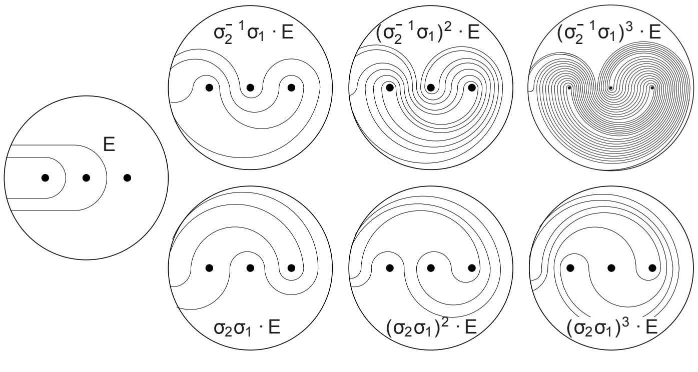
1. *$\{\tau_i\}_i$ is a Q'-quasi-geodesic in $\mathcal{T}(S)$ (meaning that, for any two points τ_{i_0} and τ_{i_1} with $|i_0 - i_1| = k$, we have ${Q'}^{-1}k - Q' \leq d_{\mathcal{T}}(\tau_{i_0}, \tau_{i_1}) \leq Q'k + Q'$);*
2. *$\{V(\tau_i)\}_i$ is a Q'-quasi-geodesic in $\mathcal{M}(S)$.*

2.4.2 Interpreting braids in terms of splitting sequences

We will now see what does this have to do with the setting of braids. This is inspired by the work of Dynnikov and Wiest, although it does not follow it strictly. First of all, we take a set of generators on B_n wider than the usual one: namely, for each $1 \leq i < j \leq n$, let Δ_{ij} be the positive permutation braid which reverses the strings placed in a position between the ith and the jth, and does not act one remaining ones. We define the Δ-*length* of a braid $b \in B_n$ as the minimum word length of b in terms of generators Δ_{ij} (this definiton is *different* from the one used by Dynnikov and Wiest), and denote it as $\delta(b)$.

The Δ-length of a braid can be given an interpretation in terms of length of a sliding and splitting sequence of train tracks. Similarly as we did at the beginning of Section 2.3, we consider a link given as the closure of a braid, and living in the solid torus so that its complement is a mapping torus over a punctured disc S. Differently from before, though, we would better consider now the braid as living in a *compact* solid torus, and therefore we add to S its external boundary, thus obtaining $\bar{S}$. The action of the braid B as a mapping class on $\bar{S}$ is completely determined if we describe its behaviour on a collection of arcs that fills it.

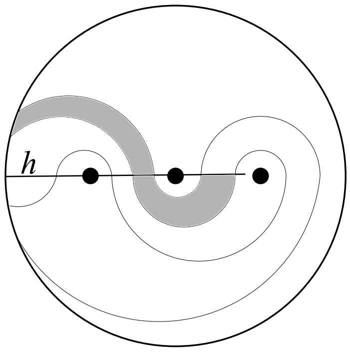
Let H be the horizontal line in $\bar{S}$ having points both endpoint on its boundary, and passing through all its punctures. We will consider the collection of arcs E in the figure below (which is taken from [DW04]): each arc of E is considered to have one end above and one below H, and to cross H only once; E consists of $n-1$ arcs in total, each of which runs between a different couple of contiguous punctures of $\bar{S}$.



We will use the term *lamination* for a collection of curves obtained from E by applying some homeomorphism of $\bar{S}$. If L' is a lamination obtained from another one L by applying a braid b, we will write $L' = b \cdot L$. Note that, if $b \cdot E = b' \cdot E$, then $b = b'$ in B_n, because two mapping classes showing the same behaviour on a set of arcs that fills $\bar{S}$ must be the same. We will define two constructions below, for which we will always suppose that laminations are isotoped to reach the minimal number of intersections with H. In order for our constructions to make sense, isotopies are allowed to move the endpoints of the components of L along $\bar{S}$, *but these are not allowed to cross* H.

The first construction is arranging its arcs into strips. Namely, take some segment h of H which includes its left endpoint, and cut all the arcs of L at their intersections with h. We now have a collection of smaller arcs. Consider homotopies of arcs that include the possibility of moving endpoints along $\partial\bar{S}$ and along $h \setminus \{\text{endpoints and punctures}\}$: call two arcs *parallel* if they are homotopic under these restrictions. Note that, between any couple of parallel arcs, no one not parallel to them is allowed. We define a ***strip*** to be a maximal collection of parallel arcs. The lamination L can therefore be regarded as a collection of strips, joined with each other along h.

Every time there are two strips, including the same number of arcs, and these arcs are pairwise attached at one end, we join them and regard the result as only one strip. For instance the two grey strips in the picture below, including 2 arcs each, are regarded as just one:



If, among the joined ends of strips, there are also the ones located at the rightmost position of h, we delete from h the corresponding rightmost subsegment. We repeat this shortening of h, until the two ends of strips at its rightmost will not result to be joined anymore; what remains of the original h at the end of this procedure will be called the ***cutting segment*** for our ***strip decomposition***. We define the ***length*** of a strip as the number of components of $\alpha \setminus H$, for α any arc in the strip. Note that, if $h = \varnothing$, then the only lamination inducing strips whose lengths are all ≤ 3 is E.

We call a ***generalized strip decomposition*** a construction that works exactly as above, except that, at the beginning, we do not suppose the strips to be *maximal* collections of parallel arcs. In other words, we may split each such collection into 2 or more sub-collections of contiguous parallel arcs; we call these chosen sub-collections strips as well. We then continue with the construction in the same way specified above. See picture below for an example.

Both regular and generalized strip decompositions also allow definition of a (sort of) train track on $\bar{S}$ that carries the lamination L. In our context, we will allow train tracks to 'have ends on the boundary of $\bar{S}$': in other words, we will re-define a pretrack on $\bar{S}$ as a topological graph, properly embedded in S in such a way that

- the intersection of the graph with $\partial\bar{S}$ consist of its 1-valent vertices, and there are no 2-valent ones;
- each edge (which we will call *branch*) is embedded smoothly;
- for each vertex x of valence > 2 (called *switch*) of the graph, there is a $v_x \in T_x\bar{S}$ such that all edges having x as endpoint have tangent direction there equal to v_x, except for one whose tangent is $-v_x$;
- for each branch of τ there is a smooth immersion $\mathbb{R} \to S$ with image contained in τ, or a smooth immersion $I \to \bar{S}$ with endpoints in $\partial\bar{S}$, passing through that branch.

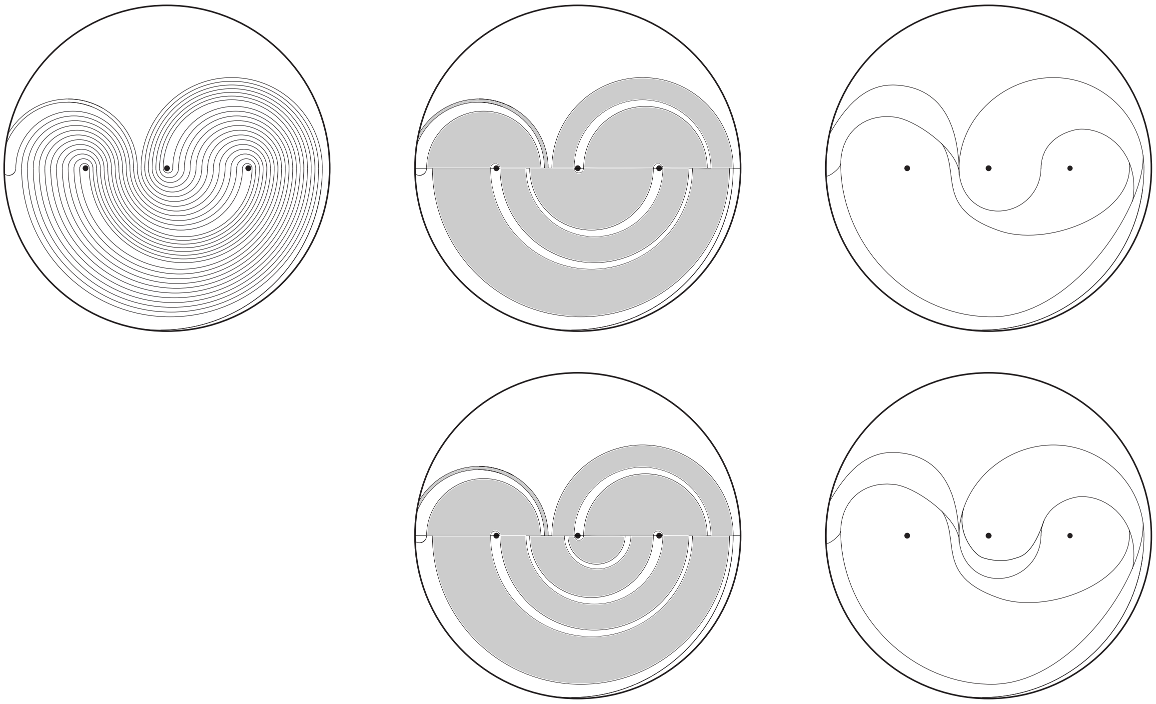
The further conditions that a pseudo-pretrack as above must satisfy in order to be considered a train track (or, to better say, a pseudo-train track) can be taken to be the same as before. Also the other definitions we have given before must be adapted. First of all, vertices of valency 1 will be not be considered as switches. Removal of the 3-valency condition on switches means that a large end can correspond to more than 2 small ones; therefore the notion of transverse measure must be generalised accordingly. The branches having and endpoint on a 1-valent vertex can be considered to be small or large according to the nature of their other end (this means, in particular, that it makes sense to perform a split move on a large one). Rather than vertex curves, we may talk now about vertex curves and arcs. Note that laminations are admitted as a particular case of pseudo-train tracks with no switches, according to this generalised definition.

Let's see now how to get a standard pseudo-pretrack as described above from a (generalized) strip decomposition of a lamination. Let's draw a graph on $\bar{S}$, first of all: for each strip in the decomposition, we pick one of the parallel arcs it includes; then we remove a small open neighbourhood of its intersection(s) with H. We get a collection A of disconnected edges. Now, for any couple of ends of strips on opposite sides of H that overlap (even if only partially), draw an edge between the corresponding vertices of the edges in A: call B this new collection of edges. Finally, turn this graph into a pseudo-pretrack by imposing that, every time two ends a_1, a_2 of edges in A are joined by an edge $b \in B$, the union of b with neighbourhoods of a_1 and a_2 is a smooth immersion of I in $\bar{S}$. It is easily checked that:

- the result is a pseudo-pretrack τ which is not in general a pseudo-train track, but it is if we started from a proper strip decomposition;

- it carries the lamination L we have started with; actually, $L = V(\tau)$.

The picture below shows an example of lamination accompanied by its strip decomposition with respect to $h = H$, and the associated pseudo-train track; the row below shows a generalized strip decomposition and the pseudo-pretrack obtained from it.



For a lamination L, call $\tau(L)$ the pseudo-train track obtained from the regular strip decomposition of L using the procedure above, for $h = H$. In the same way, if σ is some generalized strip decomposition, call $\tau(\sigma)$ the corresponding pseudo-pretrack. Note that the definition of $\tau(L)$ excludes the possibility of having 'parallel' branches, therefore there is a finite quantity of combinatorial possibilities i.e. there is a finite number of pseudo-train tracks $\tau(L)$ that can be obtained by varying the lamination L.

I am currently working on this statement:

Conjecture 2.4.3. Let $L = b \cdot E$ be a lamination on $\bar{S}$, and $\tau(L)$ be its associated pseudo-train track. There are constants a, k_0, k_1, only depending on $\bar{S}$ such that the following hold.

- There is a splitting and sliding sequence from $\tau(L)$ to L involving a number m of moves, with not more than a consecutive sliding moves, such that $\delta(b) \leq$

$k_0 m$.

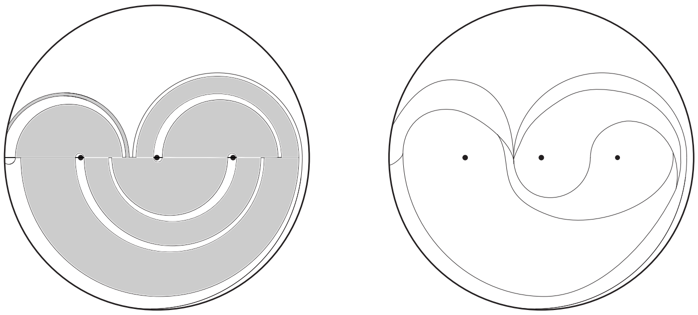
- There is a splitting and sliding sequence from $\tau(L)$ to L, whose length is $\leq k_1\delta(b)$.

The importance of this result is in that, if Theorem 2.4.2 can be applied to pseudo-train tracks as well (in particular, if we have a suitable generalization of marking graph, and of birecurrency, to cover this kind of situation), we can first of all conclude that $\delta(b)$ and $d_{\mathcal{T}}(\tau(L), L)$ are bounded by each other up to multiplicative and additive constants. Furthermore, as there are only finitely many $\tau(L)$, there is a bound, only depending on $\bar{S}$, on the distance between $\tau(L)$ and E for any L (i.e. for any b): therefore $\delta(b)$ and $d_{\mathcal{T}}(E, b \cdot E)$ are the same up to constants. All of this, if the second part of the theorem applies, will hold also for the distance in the marking graph (adapted in some way).

The result, stated this way, is not surprising at all, if we consider Proposition 2.3.9 and the subsequent remarks. But the techniques involved in this approach look suitable for a generalization to the pants graph (which is what we really want): first of all, if we re-define the pants graph by putting an edge between any couple of vertices when they have small intersection number, the result can be shown to be quasi-isometric to the regular pants graph (as it happened for the marking graph in the way described above). Therefore we can maybe try a generalization that involves *proper* train tracks rather than the modified version introduced above; then we will need a different definition for the train track graph that would result in having a natural map from it to the pants graph.

The reason why I call the statement above a conjecture is that there is some technical detail still missing. Anyway, the main points for a proof are explained below.

Let's start from the first of the two statements. First of all, we define a move on a (generalized) strip decomposition of a lamination L on $\bar{S}$. Let h be the cutting segment of the decomposition; the segments at the ends of each strip will be subsegments on h; let s, t the rightmost ones, naming them in such a way that s is shorter then t. A ***transmission*** on the (generalized) strip decomposition consists in deleting s from h and change the g.s.d. subsequently. In other words, we split t into 2 segments t_1 and t_2, with the latter exactly overlapping with s; split the strip to which t belongs into 2 parallel strips accordingly; glue together t_2 and s to get a unique strip. The picture below shows the result of a transmission effected on the (proper) strip decomposition depicted in the picture above, and the relative behaviour on a train track level; note that the segment h has been shortened accordingly.



Every time we perform a transmission, we get a new strip that might be quite long. Given a (generalized) strip decomposition σ, call a ***relaxation of*** σ a self-homeomorphism ϕ of $\bar{S}$ such that:

- if A is the set of the endpoints of the arcs of the strips of σ, the lamination $\phi(L)$ can be isotoped so to have minimal intersection with H, and $\phi(A)$ lies on H again (therefore $\phi(\sigma)$ is a strip decomposition of $\phi(L)$, performed with respect to some new cutting segment);
- strips of $\phi(\sigma)$ will have length ≤ 3.

A (generalized) strip decomposition whose strips have all length ≤ 3 will be called *relaxed*.

My claim now is that

Every generalized strip decomposition can be relaxed. In particular there exists a constant K, only depending on $\bar{S}$, such that: a regular strip decomposition σ will be relaxed by some homeomorphism whose Δ-length is $\leq K$; and the same is true for a generalized strip decomposition which is the result of a transmission operated on some relaxed *generalized strip decomposition.*

The first of the two bounds claimed is reasonable because regular strip decompositions are subject to an uniformly bounded combinatorics; the second one because strips after a transmission on a relaxed g.s.d. have length ≤ 6. If this claim is correct, we can argue as follows: first of all, perform the regular strip decomposition σ of the lamination L with respect to the whole horizontal line H. Relax it: this will involve a diffeomorphism of $\bar{S}$, corresponding to some braid $b_0 \in B_n$ with $\delta(b_0) \leq K$. Call σ_0 the strip decomposition of $b_0 \cdot L$ obtained this way. Now we define inductively σ_{j+1} as the strip decomposition obtained by taking σ_j, performing a transmission, and relaxing it according to a braid b_{j+1} with $\delta(b_{j+1}) \leq K$ (if relaxation is not needed, take $b_{j+1} = 1$).

This process will stop at some generalized strip decomposition σ_N whose cutting segment is empty. Since this decomposition is relaxed, it must be the trivial decomposition of the lamination E. Since $E = b_N \cdots b_0 \cdot L$, we must have $b_N \cdots b_0 = b^{-1}$ i.e. $\delta(b) = \delta(b^{-1}) \leq KN$.

On the other hand, consider what happens on the level of pseudo-train tracks. A transmission on a (generalized) strip decomposition σ_j results in effecting *exactly one* split or slide move on $\tau(\sigma_j)$. Note that, in particular, a pseudo-pretrack which is actually a pseudo-train track remains so under such a move, so we never actually need to consider pseudo-pretracks that are not pseudo-train tracks. Call τ_{j+1} the new pseudo-train track, and note that $\tau_{j+1} = b_{j+1}^{-1}\tau(\sigma_{j+1})$.

Therefore the sequence of train tracks

$$\tau(L) = b_0^{-1}\left(\tau(\sigma_0)\right), (b_1 b_0)^{-1}\left(\tau(\sigma_1)\right), \ldots, (b_N \cdots b_0)^{-1}\left(\tau(\sigma_N)\right) = b \cdot E = L$$

is a splitting and sliding sequence, of length $N+1$. The claim about the bounded number of consecutive slides follows from the fact that transmissions spiraling around a fixed set of punctures in $\bar{S}$ cannot induce 2 consecutive slides: therefore a long sequence of slides in our setting will force a switch to move leftwards, and there is an universal bound on the number of times this can happen. This concludes the sketch of proof of the first part of my conjecture.

The second part is simpler. Fix a decomposition $b = \delta_N \ldots \delta_1$ where each δ_j is some of the generators Δ_{ik} and $N = \delta(b)$. We claim that

There exists a constant K, depending only on $\bar{S}$, such that, for each j, a folding *and sliding sequence of length $\leq K$ connects $\delta_{j+1}\left(\tau(\delta_j \cdots \delta_1 \cdot E)\right)$ with $\delta_{j+1}\delta_j \cdots \delta_1 \cdot E$*

Recall, indeed, that each generator Δ_{ij} can be interpreted as a half-twist on $\bar{S}$ around the punctures from the ith to the jth. Since there are universal bounds on the combinatorics of a standard induced pseudo-train track of a lamination, a half-twist upon it can generate only a uniformly bounded number of portions of branches that need to be folded or slid in order to get a standard pseudo-train track again. Therefore, if we consider the sequence

$$L = \delta_N \cdots \delta_1\left(\tau(E)\right), \delta_N \cdots \delta_2\left(\tau(\delta_1 \cdot E)\right), \ldots, \tau(\delta_N \cdots \delta_1 \cdot E) = \tau(L)$$

each two consecutive terms are connected by at most K folding and sliding moves. The inverse path will therefore consist of at most KN splittings and slidings.

Chapter 3

Other possible approaches

3.1 A surface in the link complement

3.1.1 The work of Agol and Lackenby

A common way of investigating 3-manifold is cutting them along a surface. To keep the exposition easy, I will only focus on the case of 3-manifolds which are link complements.

The construction is the following. Let K be a link in S^3, and let S be an essential, properly embedded surface in $M \coloneqq S^3 \setminus K$. Let $\mathcal{N}(S)$ be an open regular neighbourhood of S, and denote $M \setminus\setminus S \coloneqq M - \mathcal{N}(S)$. Then part of the boundary of $M \setminus\setminus S$ will be naturally distinguished, namely its subsurface $P \coloneqq \partial M \cap \partial(M \setminus\setminus S)$: it is what remains of $\partial\mathcal{N}(K)$, and consists of a collection of annuli. It is called the *parabolic locus* of $M \setminus\setminus S$. Then we may look to the JSJ decomposition of $M \setminus\setminus S$, but *we will always perform it with the convention that P is removed from the boundary of $M \setminus\setminus S$*. Unless explicitly stated, this convention will be always used in the following. The reason for such a construction is the following theorem of Agol ([Ago], improved in [ADST05]).

Theorem 3.1.1. *In the setting described above, if the interior of M admits a (complete, finite volume) hyperbolic structure, then*

$$\mathrm{vol}(M) \geq -v_{oct}\chi\left(\mathrm{guts}(M \setminus\setminus S)\right)$$

where v_{oct} is the volume of the regular ideal octahedron in $\mathbb{H}^3$.

Agol's proof uses the fact that $\mathrm{vol}(M)$ is, up to a v_3 factor, also the Gromov norm of M. After using an alternative definition of Gromov norm (Agol introduces a holomogy that considers polyhedra with triangular faces, rather than simplices), he describes a way of obtaining a chain that represents the fundamental class of guts($M \backslash\backslash S$), starting from one that gives the fundamental class of M. Then he gives an upper bound the Gromov norm of the new chain in terms of the norm of the old one (i.e. the desired volume). But the norm of the new chain also admits an interpretation in terms of the Euler characteristic of the boundary of the guts and therefore the one of the guts themselves.

The best known volume estimate that uses Agol's result is the one of Lackenby about alternating link diagrams ([Lac04]):

Theorem 3.1.2. *Let D be a prime alternating diagram of a hyperbolic link K in S^3. Then*

$$v_3(t(D)-2)/2 \leq \mathrm{vol}(S^3 \setminus K) < 10v_3(t(D)-1)$$

where v_3 is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$.

Here, $t(D)$ is the ***twist number*** of the diagram D. A ***twist region*** in a link diagram is either a single crossing that does not belong to the boundary of any bigon region of the complement of the diagram; or a maximal row of bigon regions of the complement of the diagram. The twist number is the number of twist regions.

The upper bound (in the statement above it has been expressed with the constants improved by Agol and Thurston) follows from an argument that can be applied to *all* link diagrams: the twist regions are ***augmented*** by adding a circle component around each of them: this way we can simplify the twist regions without changing the homeomorphism class of the link complement; eventually, each of those will end up contributing at most 6 crossings.

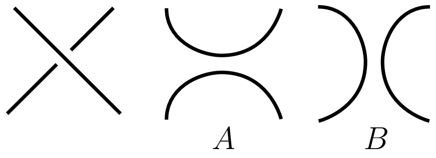
Then the complement can be divided into a number of *ideal tetrahedra* which is bounded by the crossing number of the new diagram. This subdivision is, a priori, just topological: namely, let M be the complement of an open neighbourhood of the augmented link, and crush each boundary component of M to a point: in so doing we obtain a singular manifold $\tilde{M}$. Subdividing the interior of M into ***ideal tetrahedra*** means for us subdividing $\tilde{M}$ into singular tetrahedra whose intersection with $\tilde{M}$ are exactly their vertices.

The lower bound, instead, comes from application of Agol's result: Lackenby considers a chequerboard colouring of the link diagram and connects all the white regions by gluing twisted bands at the crossings: in so doing a surface W is obtained, bounded by the link diagram. In the same way a black surface B is created. As a consequence of the diagram being prime and alternating, these surfaces are both shown to be incompressible. The remainder of the proof is a rather deep analysis of the link diagram and its complement, whose outcome is that the characteristic manifold of $S^3 \backslash\backslash B$ only consists of Seifert-fibred components, therefore the Euler characteristic of the guts coincides with the one of $S^3 \backslash\backslash B$ itself; and the latter can be understood from the diagram.

3.1.2 A machinery for computing guts

Application of Agol's theorem to positive braids is not straightforward. The first problem one encounters is that the most natural surface one might think of cutting along is the one obtained from the braid diagram via Seifert's algorithm. This will not give any restriction on hyperbolic volume: since the braid complement is a mapping torus having that surface as its base space, cutting along it will result in an I-bundle, so there will be no guts at all.

A big help here is given by Futer, Kalfagianni and Purcell in their monograph [FKP11]. Their main concern are surfaces arising from state resolutions of a link diagram. Namely, let D be a link diagram. For each crossing we consider these 2 possibilities of *resolving* it:



A choice of a resolution of each crossing is called a ***state***; we denote A (and B, resp.) the states where all crossings are resolved in the way denoted by the letter A (B, resp.) in the picture above. If σ is a state, we denote D_σ the result of resolving D according to σ; it is a collection of circles in the plane. Two graphs are naturally associated to it: the graph H_σ obtained from D_σ by connecting its circles with segments replacing the former crossings; and $\mathbb{G}_\sigma$, obtained from H_σ by crushing each circle of D_σ to a point. Furthermore, another graph $\mathbb{G}'_\sigma$ can be considered, obtained from $\mathbb{G}_\sigma$ by replacing each family of edges that connects the same two vertices with a single one.

A state σ is called ***homogeneous*** if, for each connected component of the complement of D_σ, the crossings of D that lie in that component have all been resolved in the A-way or the B-way. Of course the A state and the B state are both homogeneous.

A state σ is ***adequate*** if $\mathbb{G}_\sigma$ has no one-edge loops. D is called A- (resp. B-) ***adequate*** if the A state (resp. the B state) is adequate.

Each state σ comes with a natural associated surface S_σ bounded by the link: namely the one obtained from taking a collection of disjoint discs bounded by the circles of D_σ, and connecting them via twisted bands that reproduce the original crossing in D.

The authors report a result by Ozawa [Oza11]:

Theorem 3.1.3. *Let D be a connected diagram of a link K, such that D is σ-homogeneous for some state σ. Then the surface S_σ is essential in $S^3 \setminus K$ if and only if D is σ-adequate.*

But, more importantly, in the monograph it is described, for a homogeneous state σ, how to decompose $(S^3 \backslash\backslash S_\sigma) - P$ into *ideal polyhedra*, i.e., similarly as in Lackenby's paper (but in a more involved way), they consider the singular 3-manifold obtained from $S^3 \backslash\backslash S_\sigma$ crushing each component of P to a point, and subdivide it into polyhedra whose vertices are the crushed components of P. The decomposition they give is canonical and has the property that, if $\partial(S^3 \backslash\backslash S_\sigma) - P$ is coloured in black and the new faces of the polyhedra are coloured in white, the boundary of each polyhedron is chequerboard-coloured. Moreover the portion of $(S^3 \backslash\backslash S_\sigma) - P$ that lies above the projection plane is spanned by only 1 polyhedron.

Thanks to the properties of this decomposition the following result is proved, that allows telling whether S_σ is a fibre surface from the graph $\mathbb{G}'_\sigma$:

Theorem 3.1.4. *Let D be a diagram of a link K, and let S_σ be the state surface of a homogeneous state σ. Then the following are equivalent:*

1. *The graph $\mathbb{G}'_\sigma$ is a tree.*
2. *$S^3 \setminus K$ fibers over S^1 with fiber S_σ.*
3. *$S^3 \backslash\backslash S_\sigma$ is an I-bundle over S_σ.*

The core of the monograph is a more precise result, that computes the Euler characteristic of the guts in terms of the graph $\mathbb{G}'_\sigma$, but also of the part of the characteristic submanifold that lies in the upper polyhedron. The idea is the following: among the components of the characteristic submanifold, the Seifert-fibred pieces can be neglected as they do not affect the Euler characteristic. The I-bundles do, but they can be decomposed along discs so that each piece will have Euler characteristic 0 or 1.

More precisely, an *essential product disc* E in $S^3 \setminus\setminus S_\sigma$ is a properly embedded essential disc such that $\partial E \cap P$ consists of exactly 2 arcs. A surface T in $S^3 \setminus\setminus S_\sigma$ is *parabolically compressible* if there exists a disc C, properly embedded in $S^3 \setminus\setminus S_\sigma$, such that $C \cap T$ and $C \cap P$ each consist of an arc of ∂C, and these two arcs are disjoint from each other; and, moreover, C cannot be isotoped into T with an isotopy that keeps $C \cap P$ within P.

It can be proved that, for each I-bundle component of the characteristic submanifold of $S^3 \setminus\setminus S_\sigma$, there exists a collection of disjoint essential product discs in $S^3 \setminus\setminus S_\sigma$, each of which is contained in only one polyhedron, that cut the I bundle into prisms (i.e. I-bundles over polygons) and solid tori (i.e. I-bundles over annuli or Möbius bands). We say that such a collection *spans* the original I-bundle.

We distinguish three types of essential product disc: the *simple* ones are the ones that bound a regular neighbourhood of a white bigon face of a polyhedron of the decomposition, the *semi-simple* ones are the ones that are not themselves simple, but a finite sequence of parabolic compressions turns them into a collection of simple essential product discs; the ones that are neither simple nor semi-simple are called *complex*. We denote e_c the minimum number of complex essential product discs that are required to span the portions of I-bundles lying in the upper polyhedron. We can now state the core result of [FKP11]

Theorem 3.1.5. *Let D be a link diagram, and let σ be a state that is homogeneous and adequate for D. For X a CW-complex whose connected components are $X_1, \ldots, X_n$, denote $\chi_-(X) := \sum_{i=1}^n \max\{0, -\chi(X_i)\}$. Then*

$$\chi_-\left(\mathrm{guts}(S^3 \setminus\setminus S_\sigma)\right) = \chi_-(\mathbb{G}'_\sigma) - e_c.$$

The authors also give ways of investigating the properties, and the cardinality, of spanning collections. In particular they prove that, when D is a prime, A-adequate diagram such that, for each 2-edge loop in $\mathbb{G}_A$, its edges belong to the same twist region, then $e_c = 0$ for the polyhedral decomposition of $S^3 \setminus\setminus S_A$.

Thanks to this machinery, they give a partial answer to the problem of finding the volume of complements of positive braids. Indeed, when D is a positive braid diagram *such that each twist region has at least 3 crossings*, then all the hypotheses above for having $e_c = 0$ are verified. Moreover, it is only a combinatorial computation finding out that $\chi(\mathbb{G}'_A) \leq -2t(D)/3$. Hence, including the upper bound for generic link diagrams:

Theorem 3.1.6. *Let D be a positive braid diagram of a link K whose complement in S^3 is hyperbolic. Suppose that each twist region in D consists of at least 3 crossings. Then*

$$\frac{2v_{oct}}{3}t(D) \leq \mathrm{vol}(S^3 \setminus K) < 10v_3(t(D)-1).$$

This inspires, as a possible approach, trying to find an incompressible surface, bounded by the diagram; and use it for applying the results of Agol and of Futer-Kalfagianni-Purcell. Such a surface, of course, must not be a fiber surface; and, ideally, it should be possible (and easy) to build it from the diagram, even if it is not necessary for it to be a state surface. The idea, anyway, is that such a surface could be defined as a gluing of local pieces (for instance, pieces each of which corresponds to a permutation braid); and its local definition should be able to give a global control on incompressibility, and on the guts of the complement.

3.2 Bounds depending on the Garside structure

It makes sense trying to bound the volume of a positive (or *any*) braid in terms of its Garside decomposition. An upper bound, for instance, can be proved fairly easily:

Proposition 3.2.1. *Let b be a braid whose canonical decomposition as in Theorem 1.3.6 is $b = \Delta^{k_0}a_1^{k_1}\dots a_l^{k_l}$, where each positive permutation braid a_j is distinct from the ones appearing before and after it in this expression. Suppose that b represents a link K in S^3 such that the complement is hyperbolic. Then there exist contants c_0, c_1, only depending on the number n of strings, such that* $\mathrm{vol}(S^3 \setminus K) \leq c_0 l + c_1$.

The idea here is to apply the same procedure applied by Lackenby in his paper [Lac04] for obtaining an upper bound on volume. In his case, if we have a diagram D of a link K, some curves around twist regions are added to it, so that the complement of the new link $\hat{K}$ is homeomorphic to the one obtained in the same way from a diagram D' similar to D, but with no more than two crossings for each region. The complement of $\hat{K}$ turns out to be hyperbolic, and therefore its volume must be greater than the volume of the complement of K, since the latter is obtained from the former via Dehn surgery.

Here, we do the same. We will now describe a procedure to add some circles to the diagram of b so that the complement would be the same as adding the same curves to the complement of some $b' = \Delta^{\varepsilon_0}a_1^{\varepsilon_1}\dots a_l^{\varepsilon_l}$ where each ε_j is either 0 or 1. So, for each l we have only finitely many braids to examine. We will then give an upper bound for the Gromov norm of the complement of these links in terms of l and some constants; this will also be an upper bound for the original link K.

Proof. Rather than considering general braids, we will consider only positive braids for a simpler exposition (but there are no actual differences). Moreover, we do not treat Δ in any special way, therefore we may regard it as $\Delta = a_0$.

Step 1: For each j such that $k_j > 1$ in the decomposition of b, the permutation underlying a_j consists of disjoint transpositions.

Consider the geometrical interpretation of the Elrifai-Morton algorithm ([EM94], pp. 13–14). The canonical decomposition of a braid is such that, if two strings meet in some factor of the canonical decomposition, they also meet in the ones preceding it. Since the portion of braid decomposition $a_j^{k_j}$ consists of permutation braids that are all equal to each other, in particular they all have the same number of crossings. Therefore it is impossible that two strings cross in the i-th factor of these k_j ones, but not in the $(i+1)$-th. This means that, in each repetition of the factor a_j in this sequence, the strings that cross each other are always the same. Let π_j be the permutation of the strings underlying the permutation braid a_j. According to Lemma 2.4 in the paper by Elrifai and Morton there are 2 possibilites for two strings that, before applying $a_j^{k_j}$ occupy the x-th and the y-th positions respectively, with $x < y$:

- the strings cross in the first a_j (and, therefore, in *all* the ones of the sequence). In this case $\pi_j(x) > \pi_j(y)$ and, reiterating, $\pi_j^2(x) < \pi_j^2(y)$;
- the strings do not cross in the first a_j (and, therefore, in *any* the ones of the sequence). In this case $\pi_j(x) < \pi_j(y)$ and, reiterating, $\pi_j^2(x) < \pi_j^2(y)$.

This means that, for each $x, y \in \{1, \ldots, n\}$ we have $x < y \Rightarrow \pi_j^2(x) < \pi_j^2(y)$, i.e. π_j^2 is the identity. This can only occur if π_j can be written as a product of disjoint transpositions.

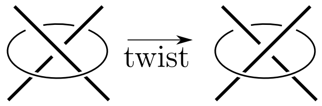
Step 2: A factor a_j as above is such that $\mathrm{rev}(a_j) = a_j$, and $\tau(a_j) = a_j^{-1}$ as words in the braid group.

Here we mean the notation used by Elrifai and Morton i.e. $\mathrm{rev}(a_j)$ is the braid obtained from a_j by reversing the order of the crossings (i.e. rotating the picture of π around a horizontal axis); and $\tau(a_j)$ is obtained from a_j by replacing each positive crossing with a negative one.

For the first claim: a_j (without the underpassing/overpassing information at each crossing) can be pictured as a superposition of X-shaped pictures, each one corresponding to a transposition. It is quite natural that reversing the braid will not change the picture. Also, each crossing must be positive both before and after reversing, to the whole diagram of this portion stays the same. The second claim follows from the general fact that $a_j^{-1} = \tau(\mathrm{rev}(a_j)) = \tau(a_j)$.

Step 3: For each $j = 0, \ldots, l$ there is a collection $\mathcal{C}_j$ of disjoint curves in S^3, each of which bound a horizontal disc on the top of the factor $a_j^{k_j}$, transversal to the braid diagram, such that $S^3 \setminus (K \cup \mathcal{C}_j)$ is homeomorphic to $S^3 \setminus (K' \cup \mathcal{C}_j)$, where K' is the link corresponding to a braid $b' = a_0^{k_0} \ldots a_j^{\varepsilon_j} \ldots a_l^{k_l}$ with $\varepsilon_j = 0$ or 1.

We may suppose $k_j > 1$. Take a circle around each crossing of the uppermost copy of a_j: let $\mathcal{C}_j$ be this collection. We claim that the complement of $K \cup \mathcal{C}_j$ in S^3 is the same as the complement of $K_1 \cup \mathcal{C}_j$, where K_1 is the link corresponding to the braid $b_1 = a_0^{k_0} \ldots \tau(a_j) a_j^{k_j - 1} \ldots a_l^{k_l}$. Indeed, for each crossing in the uppermost copy of a_j we can reverse the crossing from positive to negative: cut along a disc bounded by the circle that surrounds the crossings; twist the lower face by 2π; glue again. This is exemplified by the following picture:



For a better understanding of what follows, though, is it better to isotop the collection $\mathcal{C}_j$ on top of the first a_j factor. This will not affect the result of twisting around them, but this way we may regard each element in $\mathcal{C}_j$ as a curve in S^3 that stays fixed under the tranformations of the link diagram we perform. Now, as a consequence of the previous step, we have $b_1 = a_0^{k_0} \ldots a_j^{k_j - 2} \ldots a_l^{k_l}$. If $k_j > 3$, we can reiterate this step; we do not have to select new curves around the crossings of new top factor of $a_j^{k_j - 2}$, because they are exactly the ones of the previous collection $\mathcal{C}$. The procedure can be reiterated $\lfloor k_j/2 \rfloor$ times, until the original factor $a_j^{k_j}$ is replaced by a simple a_j or the trivial braid.

Step 4: We will now try to find a bound on the number of ideal tetrahedra required for a simplicial subdivision of the interior of $M := S^3 \setminus \left(K \cup \bigcup_j \mathcal{C}_j\right)$. This will result in an upper bound on its Gromov norm. Indeed, if $\tilde{M}$ is obtained, as described in the previous section, by crushing each component of ∂M to a point, there is a canonical isomorphism $H_3(M, \partial M; \mathbb{R}) \cong H_3(\tilde{M}, \text{singular locus}; \mathbb{R})$.

Thanks to the previous step we can suppose that all $k_j = 1$ and that the number of factors is, rather than the original l, some $l' \leq l$. This will be enough in order to prove the claim for *any* K.

First of all, let's picture $K \cup \bigcup_j \mathcal{C}_j$ as a diagram of the braid b plus the circles of $\mathcal{C}_j$ on top of each factor a_j. This diagram will have not more that $\binom{n}{2}$ crossings coming from a_j (because it is a permutation braid: two strings do not intersect more than once), and each circle of $\mathcal{C}_j$ (remember that they are circles around the crossings, pushed on top of the permutation factor) cannot be involved in more than $2n$ crossings. So, the diagram is divided into l' portions, each of which contributes a number of crossings only depending on n, which we call N.

Now, the bound on the number of tetrahedra is exactly the same as the one given by Lackenby. The selected diagram of $K \cup \bigcup_j \mathcal{C}_j$ can be regarded (without its over- and underpassing information) as a 4-valent graph lying on S^2. Note that, differently from the case of alternating links, there is not necessarily an isomorphism between this graph and the graphs induced on the 2 polyhedral cells of the canonical decomposition described by Benedetti-Petronio ([BP92]) and Lackenby. But the complementary faces of the diagram still have a natural one-to-one correspondence with the faces of each of the two cells; and this correspondence preserves the number of edges of the face. Therefore in the following computations we will refer to the numbers of faces, edges of a face f, all edges, vertices respectively in *any* of the link graph or the cell boundary, with F, $E(f)$, E, and V. We have proved above that $V \leq Nl'$.

Considering that the link diagram is a 4-valent graph, we have $4V = 2E$ i.e. $E = 2V$. Moreover $2 = \chi(S^2) = F - E + V = F - V$, therefore $F \leq Nl' + 2$.

We now subdivide the boundary of each of the 2 cells above into triangles. Bigon faces will be crushed to an only edge, the other ones will be divided into $E(f) - 2$ triangles each. The total number of triangles for each cell boundary will therefore be $T = \sum_f (E(f) - 2) = 2(E - F) = 2(2V - V - 2) = 2V - 4 \leq 2Nl' - 4$.

This triangulation can be completed to a simplicial subdivision of the whole 2-cells: starting from a selected vertex of a 2-cell, we 'cone' on each triangle of its boundary from this vertex. This results in each cell being divided into less than T triangles. Therefore, there are constants c_0, c_1 such that the interior of $S^3 \setminus \left(K \cup \bigcup_j \mathcal{C}_j\right)$ is split into not more than $c_0 l' + c_1$ tetrahedra.

Step 5: Conclusion

We have the following chain of inequalities:

$$\mathrm{vol}\left(S^3 - K\right) = v_3 \|[S^3 \setminus K, \partial(S^3 \setminus K)]\| \leq v_3 \|[M, \partial M]\| \leq v_3(c_0 l - c_1)$$

where the first equality is due to Theorem 1.2.9; the following inequality is due to

Proposition 1.2.7 and the remarks in the same paragraph; the last inequality is due to Step 4 above. □

REMARK 3.2.2. This result is comparable to the upper bound in terms of number of twists. Indeed, if a positive braid has an expression $b = \sigma_{\alpha_1}^{\beta_1} \dots \sigma_{\alpha_j}^{\beta_j} \dots \sigma_{\alpha_m}^{\beta_m}$ (with all $\beta_j \geq 3$) then the twist number is m, whereas a possibility for dividing it into permutation braids is $b = \sigma_{\alpha_1}^{\beta_1 - 1}(\sigma_{\alpha_1}\sigma_{\alpha_2}) \dots (\sigma_{\alpha_{j-1}}\sigma_{\alpha_j})\sigma_{\alpha_j}^{\beta_j - 2}(\sigma_{\alpha_j}\sigma_{\alpha_{j+1}}) \dots (\sigma_{\alpha_{m-1}}\sigma_{\alpha_m})\sigma_{\alpha_m}^{\beta_m - 1}$. This decomposition involves $2m-1$ factors, each elevated to some power, and the canonical one can only be shorter. ◊

Bibliography

[ADST05] I. Agol, N. M. Dunfield, P. A. Storm, and W. P. Thurston. Lower bounds on volumes of hyperbolic Haken 3-manifolds. *ArXiv Mathematics e-prints*, June 2005.

[Ago] Ian Agol. Lower bounds on volumes of hyperbolic haken 3-manifolds, preprint. *arXiv preprint math.GT/9906182*.

[Ber60] Lipman Bers. Simultaneous uniformization. *Bull. Amer. Math. Soc.*, 66:94–97, 1960.

[BH96] Steven A. Bleiler and Craig D. Hodgson. Spherical space forms and Dehn filling. *Topology*, 35(3):809 – 833, 1996.

[Bon02] Francis Bonahon. Geometric structures on 3-manifolds. *Handbook of geometric topology*, 93164, 2002.

[BP92] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Springer, 1992.

[Bro01] Jeffrey F. Brock. Weil-Petersson translation distance and volumes of mapping tori. *ArXiv Mathematics e-prints*, September 2001.

[Bro03] Jeffrey F. Brock. The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores. *J. Amer. Math. Soc.*, 16:495–535, 2003.

[Cro93] Peter R Cromwell. Positive braids are visually prime. *Proceedings of the London Mathematical Society*, 3(2):384–424, 1993.

[CZ06] Huai-Dong Cao and Xi-Ping Zhu. A complete proof of the Poincaré and geometrization conjectures-application of the Hamilton-Perelman theory of the Ricci flow. *Asian Journal of Mathematics*, 10(2):165–492, 2006.

[DW04] Ivan Dynnikov and Bert Wiest. On the complexity of braids. *arXiv preprint math/0403177*, 2004.

[EM94] Elsayed A Elrifai and Hugh R Morton. Algorithms for positive braids. *Quarterly Journal of Mathematics*, 45(180):479–498, 1994.

[FKP08] David Futer, Efstratia Kalfagianni, and Jessica Purcell. Dehn filling, volume, and the Jones polynomial. *Journal of Differential Geometry*, 78(3):429–464, 2008.

[FKP11] David Futer, Efstratia Kalfagianni, and Jessica S Purcell. Guts of surfaces and the colored Jones polynomial. *arXiv preprint arXiv:1108.3370*, 2011.

[FM11] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton Mathematical Series. Princeton University Press, 2011.

[FS11] David Futer and Saul Schleimer. Cusp geometry of fibered 3-manifolds. *arXiv preprint arXiv:1108.5748*, 2011.

[GMW10] Juan González-Meneses and Bert Wiest. Reducible braids and Garside theory. *arXiv preprint arXiv:1008.0238*, 2010.

[Joh79] Klaus Johannson. *Homotopy Equivalence of Three-Manifolds with Boundaries*. Number 761. Springer, 1979.

[JS78] William Jaco and P Shalen. A new decomposition theorem for irreducible sufficientlylarge 3–manifolds. In *Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc*, volume 32, pages 71–84, 1978.

[Lac04] Marc Lackenby. The volume of hyperbolic alternating link complements. *Proceedings of the London Mathematical Society*, 88(1):204–224, 2004.

[Men84] William Menasco. Closed incompressible surfaces in alternating knot and link complements. *Topology*, 23(1):37–44, 1984.

[MM00] Howard A Masur and Yair N Minsky. Geometry of the complex of curves II: Hierarchical structure. *Geometric and Functional Analysis*, 10(4):902–974, 2000.

[MMS12] Howard Masur, Lee Mosher, and Saul Schleimer. On train-track splitting sequences. *Duke Mathematical Journal*, 161(9):1613–1656, 2012.

[Oza11] Makoto Ozawa. Essential state surfaces for knots and links. *Journal of the Australian Mathematical Society*, 91(03):391–404, 2011.

[Sco83] Peter Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15, 1983.

[Thu98] W. P. Thurston. Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle. *ArXiv Mathematics e-prints*, January 1998.

[TM79] William P Thurston and John Willard Milnor. *The geometry and topology of three-manifolds*. Princeton University Princeton, 1979.