Suslin's Problem and Martin Axiom

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Such a set is called a Suslin line. The existence of a Suslin line is equivalent to the existence of a normal Suslin tree.

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Suslin Tree

A tree is called a *Suslin tree* if:

1. $height(T) = \omega_1$
2. every branch in T is at most countable
3. every antichain in T is at most countable

A Suslin tree is called *normal* if:

1. T has a unique least point
2. each level of T is at most countable
3. x not maximal has infinitely many immediate successors
4. $\forall x \in T$ there is some $z > x$ at each greater level
5. if $o(x) = o(y) = \beta$ with β limit and $\{z \colon z < x\} = \{z \colon z < y\}$ then $x = y$

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Solovay-Tennenbaum

There is a model $\mathcal{M}$ of ZFC such that $\mathcal{M} \models MA + 2^{\aleph_0} > \aleph_1$.

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$$\mathcal{M}[\mathcal{G}_1][\mathcal{G}_2] = \mathcal{M}[\mathcal{G}]$$

We will define this filter using Boolean algebras.

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D is a maximal subset in $\mathcal{M}^B$ such that:

1. $||c \in \mathbf{C}|| = 1 \ \forall c \in D$
2. $c_1, c_2 \in D, c_1 \neq c_2 \Rightarrow ||c_1 = c_2|| < 1$

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I define $+_D$:

$$\forall c_1, c_2 \in D \ \exists c \in D \text{ such that } ||c = c_1 +_C c_2|| = 1$$

this c is unique and I define $c = c_1 +_D c_2$.
The operations $\cdot_D$ and $-_D$ are defined similarly.

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The operations $\cdot_D$ and $-_D$ are defined similarly.
With this operations D is a complete Boolean algebra (in $\mathcal{M}$).
I define $B * \mathbf{C} = D$.

Theorem

Let B be a complete Boolean algebra in $\mathcal{M}$, let $\mathbf{C} \in \mathcal{M}^B$ be such that $||\mathbf{C}$ is a complete Boolean algebra$|| = 1$ and let $D = B * \mathbf{C}$ such that B is a complete subalgebra of D. Then

1. If $\mathcal{G}_1$ is an $\mathcal{M}$-generic ultrafilter on B, $C = i_{\mathcal{G}_1}(\mathbf{C})$ and $\mathcal{G}_2$ is an $\mathcal{M}[\mathcal{G}_1]$-generic ultrafilter on C then there is an $\mathcal{M}$-generic ultrafilter $\mathcal{G}$ on $B * \mathbf{C}$ such that:
$$\mathcal{M}[\mathcal{G}_1][\mathcal{G}_2] = \mathcal{M}[\mathcal{G}]$$
2. If $\mathcal{G}$ is an $\mathcal{M}$-generic ultrafilter on $B * \mathbf{C}$. $\mathcal{G}_1 = \mathcal{G} \cap B$ and $C = i_{\mathcal{G}_1}(\mathbf{C})$ then there is an $\mathcal{M}[\mathcal{G}_1]$-generic ultrafilter $\mathcal{G}_2$ on C such that:
$$\mathcal{M}[\mathcal{G}_1][\mathcal{G}_2] = \mathcal{M}[\mathcal{G}]$$

Lemma

B satisfies ccc and $||\mathbf{C}$ satisfies ccc $|| = 1$ iff $B * \mathbf{C}$ satisfies ccc.

Let α be a limit ordinal.
Let $\{B_i\}_{i<\alpha}$ a sequence such that

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Direct limit

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Lemma

Then if each B_i is k-saturated then B is k-saturated.
In particular if each B_i satisfies ccc then B satisfies ccc.

Let $\mathcal{M}$ be a transitive model of ZFC + GCH.
We will construct a complete Boolean algebra B such that if $\mathcal{G}$ is an $\mathcal{M}$-generic filter on B then

$$\mathcal{M}[\mathcal{G}] \models MA + 2^{\aleph_0} \leq \aleph_2$$

B_α

Let $\{B_\alpha\}$ be a sequence such that:

1. $\alpha < \beta \Rightarrow B_\alpha$ is a complete subalgebra of B_β
2. γ limit $\Rightarrow B_\gamma = limdir_{i \leq \gamma} B_i$
3. each B_α satisfies ccc
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So we have:

1. B satisfies ccc
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$\mathcal{M}[\mathcal{G}]$ perserves cardinals and $\mathcal{M}[\mathcal{G}] \models 2^{\aleph_0} \leq \aleph_2$ (Jech, lemma 19.4).

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Let $\mathbf{C} \in \mathcal{M}^{B_\alpha}$ be the complete Boolean algebra such that:

- $||\mathbf{C}$ is the trivial algebra $|| = -b$
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Now, since $\mathcal{M}[\mathcal{G}_\alpha]$ is a submodel of $\mathcal{M}[\mathcal{G}]$, we have

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Using a previous Theorem exists $\mathcal{H}$ $\mathcal{M}[\mathcal{G}_\alpha]$-generic filter on (ω_1, R) such that

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So $\mathcal{H}$ is $\mathcal{D}$-generic on (ω_1, R) and we conclude.