WEEK TEN

2016 February 29, Week 1 - Lecture 1

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Definition 1 (Generalized unions and intersection, from (9.)). Given a collection of sets $\mathcal{B} \neq \emptyset$, we define $\cap \mathcal{B}$ and $\cup \mathcal{B}$ as follows:

$$x \in \cup \mathcal{B} \iff \exists A \text{ s.t. } x \in A \in \mathcal{B}$$

$$x \in \cap \mathcal{B} \iff \forall A \in \mathcal{B} (x \in A).$$

Definition 2 (Choice functions, from (10.)). Given a set $A \neq \emptyset$, a choice function $\phi$ is a function $\phi: \mathcal{P}(A)^* \to A$ such that $\phi(B) \in B$ for every $B \in \mathcal{P}(A)^*$.

Axiom 1 (Choice Axiom, from (11.)). Every non-empty set $A$ has a choice function.

Proposition 1 (Properties of the quotient set, from (14.)). Given an equivalence relation $(A, R)$ the following properties hold:

(i) for every $H \in A/R$, the set $H$ is non-empty
(ii) $\cup A/R = A$

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(iii) given $H, G$ in $A/R$ there holds

$$(H \cap G \neq \emptyset) \Rightarrow (H = G).$$

Proof.
(i). Let $x \in A$ be such that $H = G_x$. From (R), $x \in R_x$, therefore, $H$ is non-empty.
(ii). Since $A/R$ is a collection of subsets of $A$, we have

$$\cup A/R \subseteq A.$$ 

Now, given $x \in A$, from (i), we have $x \in G_x$. From $x \in G_x \in A/R$ we have $x \in A/R$.

(iii). Let $x$ and $y$ be such that $H = R_x$ and $G = R_y$. Suppose that $H \cap G$ is non-empty and let $z \in A$ be such that $z \in R_x \cap R_y$. Let $w \in R_z$ then. From (S),

$$xRw \Rightarrow wRx.$$ 

Since $z \in R_x$, we have $xRz$ From (T),

$$(wRz) \wedge (zRx) \Rightarrow wRx.$$ 

Since $z \in R_y$, we have $zRy$. Again, from (T) and (S)

$$(wRz) \wedge (zRy) \Rightarrow wRy \Rightarrow yRw \Rightarrow w \in R_y.$$ 

By switching the role of $x$ and $y$, we obtain the reversed inclusion $R_y \subseteq R_x$. Then $R_x = R_y$, implying $G = H$. $\square$

Definition 3 (Fully orderings, from (16.)). A partial ordering $(A, R)$, is a full ordering if for every $x, y \in A$ either $xRy$ or $yRx$.

Axiom 2 (Class Construction Axiom of Zermelo-Fraenkel, from (17.)). Given a sentence $p(x)$ and a set $A$ there exists a set $S \subseteq A$ such that $x \in S \iff x \in A \wedge p(x)$.

Theorem 1 (Uniqueness up to isomorphism, from (19.)). Let $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$ two sets satisfying the field, positivity and completeness Axioms. Then there exists a bijective function $g: R_1 \to R_2$ such that

$$g(a + b) = g(a) + g(b), \quad g(ab) = g(a)g(b), \quad g(P_1) \subseteq P_2$$

where $P_1$ and $P_2$ are the positive sets.
34. \( \mathbb{N} \cap (0,1) = \emptyset \), Exercise 1
35. given \( n, m \in \mathbb{N} \) such that \( n > m \), \( n - m \in \mathbb{N} \), ex. 9, page 13 and Exercise 2
36. the set of Natural numbers is well-ordered, Theorem 1, page 11
37. subsets of \( \mathbb{N} \) bounded from above have maximum, Proposition 3
38. Archimedean property, page 11
39. integers, Definition 5
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41. dense subsets, page 12
42. the set of rational numbers is dense, page 12
43. the set \( \mathbb{N}_k \), Definition 6
44. finite sets, page 13
45. countable sets, page 13
46. the Bernstein’s Lemma, Lemma 1.

**Proposition 2.** \( (\mathbb{R}, \leq) \) is an ordered set.

*Proof.* The reflectivity holds, because \( x \leq x \Rightarrow x = x \).

(T). Given \( x \leq y \) and \( y \leq z \), we have

\[(x = y) \vee (y - x \in P)\]

and

\[(y = z) \vee (z - y \in P).\]

We have to show that \( x \leq z \), in all the four cases. If equalities hold in (1) and (2), then

\[(x = y) \wedge (y = z) \Rightarrow x = z \Rightarrow x \leq z.\]

If equality holds in (1) but not in (2), then

\[(x = y) \wedge (z - y \in P) \Rightarrow z - x \in P.\]

The case where the equality holds in (2) but not in (1) is similar. Finally, we consider the case where we have two strict inequalities:

\[(y - x \in P) \wedge (z - y \in P) \Rightarrow (z - y) + (y - x) = z - x \in P.\]

We used P1).

(A). Let \( x, y \in \mathbb{R} \) be such that \( x \leq y \wedge y \leq z \). If equality holds in one of the two inequalities, then, clearly, \( x = y \). So, we study only the case \( x < y \wedge (y < z) \). We have

\[y - x, -(y - z) \in P\]

which gives a contradiction with P2). \(\square\)

**Definition 4** (Least and greatest element, from (26.)). Let \( M \) be an upper bound for \( E \); if \( M \in E \) we call it greatest element. Similarly, a least element \( m \) for \( E \) is a lower bound which belongs to \( E \).

**Exercise 1** (from (34.)). \( \mathbb{N} \cap (0,1) \).

*Proof.* This follows from the fact that \( n \geq 1 \) for every \( n \in \mathbb{N} \), (47.). \(\square\)

**Exercise 2** (from (35.), ex. 9, page 13). Given \( n, m \in \mathbb{N} \) such that \( n > m \), \( n - m \in \mathbb{N} \).
Proof. Here I am writing a slightly different proof from the one I showed you during the lectures. We consider the property \( p(n) : n - 1 \in \mathbb{N} \) and define
\[
S := \{ n \in \mathbb{N} \mid p(n) \} \cup \{1\}
\]
and prove that \( S = \mathbb{N} \). Clearly, \( 1 \in S \), by definition. Now, suppose that \( n \in S \). Then
\[
(n + 1) - 1 = n \in \mathbb{N}.
\]
Now we consider the property \( q(m) : n > m \Rightarrow n - m \in \mathbb{N} \) and the set \( T := \{m \mid q(m)\} \). We proved that \( 1 \in T \). Now, suppose that \( m \in T \). We will show that \( m + 1 \in T \). Given \( n > m + 1 \), clearly \( n \neq 1 \). Then \( n - 1 \in \mathbb{N} \). We have
\[
n > m + 1 \Rightarrow n - 1 > m.
\]
Since \( m \in T \) and \( n - 1 \in \mathbb{N} \), \( n - (m + 1) \in \mathbb{N} \).

\[\square\]

**Proposition 3** (from (37.)). Subsets of \( \mathbb{N} \) bounded from above have a maximum.

**Proof.** Let \( E \subseteq \mathbb{N} \) be a non-empty subset bounded from above. From the Completeness Axiom there exists \( c := \sup(E) \). We prove that \( c = \max(E) \). On the contrary, we consider \( c - 1 \). Since \( c \) is the least upper bound, \( c - 1 \) is not an upper bound. Then there exists \( n \in E \) such that
\[
c - 1 < n < c.
\]
Clearly \( n \neq c \) because, otherwise, we would have \( c \in E \). Since \( n < c \), it is not an upper bound of \( E \). Then there exists \( m \in E \) such that
\[
n < m < c.
\]
We set \( k := m - n \). By Exercise 2, \( k \in \mathbb{N} \) and \( 0 < k < 1 \). Therefore, we obtained a contradiction with (34.).

**Definition 5** (Integers, from (39.)). The set of integers \( \mathbb{Z} \) is defined by the property
\[
p(m) : (m = 0) \lor (m \in \mathbb{N}) \lor (-m \in \mathbb{N}).
\]

**Definition 6** (Definition of \( \mathbb{N}_k \), from (43.)). We define \( \mathbb{N}_k \) the subset of \( \mathbb{N} \) of the natural numbers satisfying \( 1 \leq n \leq k \).

**Lemma 1** (Bernstein's Lemma, from (46.)). Given two non-empty sets \( A \) and \( B \) such that there exists \( f: A \rightarrow B \) injective and \( g: B \rightarrow A \) injective, there exists \( h: A \rightarrow B \) bijective.

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47. If \( g: A \rightarrow B \) is bijective, then \( A - \{x\} \approx B - \{g(x)\} \), Lemma 2
48. if \( k \geq 2 \), then \( \mathbb{N}_k - \{x\} \approx \mathbb{N}_{k-1} \) for every \( x \in \mathbb{N}_k \), Lemma 3
49. \( \mathbb{N}_h \approx \mathbb{N}_k \) if and only if \( h = k \), Proposition 4
50. given \( B \neq \emptyset \), if \( B \subseteq \mathbb{N} \) is not finite, then \( B \approx \mathbb{N} \), Theorem 3, page 13
51. \( \mathbb{N} \approx \mathbb{N} - \{x\} \) for every \( x \in \mathbb{N} \). Then \( \mathbb{N} \) is not finite, Proposition 5
52. if \( n = ab \), then \( n \geq a, b \)
53. \( \mathbb{N} \times \mathbb{N} \) is countable, Corollary 4, 14
54. the continuum, \( \mathbb{R} \), and functional cardinality, \( \mathcal{P}(\mathbb{R}) \).

**Lemma 2** (From (47.)). If \( A \) and \( B \) are non-empty and there exist \( g: A \rightarrow B \) bijective, then \( A - \{x\} \approx B - \{g(x)\} \), provided the two sets are non-empty.
Proof. We set \( A' := A - \{x\} \) and \( B' := B - \{g(x)\} \). We define \( g' : A' \to B' \) as \( g'(a) = a \). Firstly, \( g'(a) \in B' \), otherwise, \( g'(a) = g(x) \), hence \( a = x \) because \( g \) is injective; however, this contradicts the assumption \( a \in A' \). Clearly, \( g \) is an injective function; given \( y \in B' \), there exists \( x' \in A \) such that \( g(x') = y \). We show that \( x' \neq x \); on the contrary, \( B' \ni y = g(x') = g(x) \notin B' \) we obtain a contradiction. \( \square \)

Lemma 3 (From (48.)). If \( k \geq 2 \), then \( N_k - \{x\} \approx N_{k-1} \) for every \( x \in N_k \).

Proof. We define the function \( g : N_k - \{x\} \to N_{k-1} \) as follows: \( g(y) = y \) if \( y \leq x - 1 \), \( g(y) = x + 1 \) if \( y \geq x + 1 \).

Proposition 4 (From (49.)). Given \( h, k \in \mathbb{N} \), \( N_h \approx N_k \) if and only if \( h = k \).

Proof. If \( h = k \), then \( N_h = N_k \). Then \( N_h \approx N_k \). In order to prove the converse implication, we apply the Mathematical Induction Principle to the set \( S := \{h \in \mathbb{N} \mid \forall k( N_h \approx N_k \Rightarrow h = k)\} \).

(i) \( 1 \in S \). If \( N_1 \approx N_k \), then there exists a bijective function \( g : N_1 \to N_k \). Thus, \( g \) is surjective and
\[ N_k = g(N_1) = \{g(1)\}. \]
Since \( 1, y \in N_k \), we have \( 1 = y = g(1) \).

(ii) \( h \in S \Rightarrow h + 1 \in S \). Suppose that \( N_{h+1} \approx N_k \). Firstly, we notice that \( k \geq 2 \). In fact, if \( k = 1 \), we have
\[ N_{h+1} \approx N_1. \]
We can, then, apply the case \( 1 \in S \) and conclude that \( h + 1 = 1 \) which contradicts the assumption that \( h \in \mathbb{N} \). There exists a bijective function \( g : N_{h+1} \to N_k \). Since \( k \geq 2 \), we can apply Lemma 2 and obtain
\[ N_h = N_{h+1} - \{h + 1\} \approx N_k - \{g(h + 1)\}. \]
In fact, both sets are non-empty, because \( k \geq 2 \). Then, from Lemma 3
\[ N_k - \{g(h + 1)\} \approx N_{k-1}. \]
Then \( N_h \approx N_{k-1} \). Since \( h \in S \), we have \( h = k - 1 \). Then \( h + 1 = k \). \( \square \)

Proposition 5 (From (51.)). \( N \approx N - \{x\} \) for every \( x \in N \). Then \( N \) is not finite.

Proof. The bijective function is defined as \( g(k) = k \) if \( k \leq x - 1 \) and \( g(k) = k + 1 \) if \( k \geq x \). Now, if \( N \) was finite, then \( N \approx N_k \) for some \( k \geq 1 \). Since \( 1, 2 \in \mathbb{N} \) we can suppose that \( N - \{x\} \) is non-empty. Then, from Lemma 2 and Lemma 3, we obtain
\[ N_k \approx N \approx N - \{x\} \approx N_k - \{y\} \approx N_{k-1} \]
which implies \( k = k - 1 \), from Proposition 4 and we obtain a contradiction. \( \square \)

2016 March 17, Week 3 - Lecture 2

55. \( \mathbb{Q} \) is countable, Proposition 6
56. non-degenerate intervals in \( \mathbb{R} \) are uncountable, Theorem 7, page 15
57. \( I_r(x) := (x - r, x + r) \)
58. open sets, Definition page 16
59. union and finite intersections of open sets are open, Proposition 8
60. closure points, Definition, page 17
61. closed sets, page 17
62. convex sets, Definition 7
63. intervals are convex, Proposition 7
Proposition 6 (From (55.)). \( Q \) is countable.

Proof. Given \( q \in Q \), we define

\[
E_q := \{ q \in \mathbb{N} \mid qx \in \mathbb{Z} \}.
\]

Since \( q \) is a rational number, \( E_q \neq \emptyset \). Then, by the Well Ordering Theorem, the set has a minimum. We define

\[
h : Q \to \mathbb{Z} \times \mathbb{N}, \quad h(q) := (q \min(E_q), \min(E_q)).
\]

This function is injective: given \( q, q' \) such that \( h(q) = h(q') \), we have

\[
q \min(E_q) = q' \min(E_{q'}), \quad \min(E_q) = \min(E_{q'}).
\]

If we substitute the equality on the right in the left equality, we obtain \( q = q' \). Then, we have a chain of injective functions

\[
Q \to \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}
\]

which is injective. Thus, by (50.), \( Q \) is countable.

Definition 7 (Convex sets, from (62.)). A subset \( S \subseteq \mathbb{R} \) is convex if for every \( x \leq y \in S \), there holds \( [x, y] \subseteq S \).

Proposition 7 (Intervals are convex, from (63.)). Intervals are convex.

Proof. Let \([a, b]\) be a closed bounded interval. Given \( x, y \in [a, b] \), we have \( a \leq x \leq b \) and \( a \leq y \leq b \). Then if \( x \leq z \leq y \), we have \( a \leq x \leq z \leq y \leq b \), so \( z \in [a, b] \).

Proposition 8 (Convex sets are interval, from (64.)). Convex sets are intervals.

Proof. Now, suppose that \( S \) is convex. We divide the proof into four cases:

1. \( S \) is bounded. By the Completeness Axiom there are \( a := \inf(S) \) and \( b := \sup(S) \). Clearly, \( S \subseteq [a, b] \). Now, we show that \( (a, b) \subseteq S \). In fact, let \( x \) be an element of \( (a, b) \). Then

\[
a < x.
\]

Since \( a \) is the g.l.b., \( x \) is not a lower bound of \( S \). Therefore, there exists \( x' \in S \) such that \( x' < x \). We also have \( x < b \). Since \( b \) is the l.u.b., \( x \) is not an upper bound for \( S \). Then, there exists \( x'' \in S \) such that \( x < x'' \). Since \( S \) is convex, \( [x', x''] \subseteq S \). Since \( x' < x < x'' \), also \( x \in S \). Since \( a < x' \) and \( x'' \leq b \), then the set \( S \) satisfies

\[
(a, b) \subseteq S \subseteq [a, b].
\]

There are only four sets satisfying the inclusions above:

\[
[a, b], \quad (a, b], \quad [a, b), \quad (a, b).
\]

2. \( S \) is unbounded from below and bounded from above. We define

\[
b := \sup(S).
\]

We can prove that

\[
(-\infty, b) \subseteq S \subseteq (-\infty, b].
\]

Suppose that \( x < b \). Then \( x \) is not the l.u.b. Then, there exists \( x' \in S \) such that \( x < x' \). Since \( S \) is not bounded from below, \(-(|x| + 1) \) is not a lower bound. Then, there exists \( x'' \in S \) such that

\[
x'' < -(|x| + 1) < x < x'.
\]

Since \( S \) is convex,

\[
x \in [x', x''] \subseteq S.
\]
Then $x \in S$. Sets satisfying (3) are only

$$(-\infty, b), \quad (-\infty, b].$$

The cases (3) and (4) where $S$ is unbounded from above and bounded from below, or $S$ is unbounded from below and from above, are similar to the case (2). 

2016, March 21 - Week 4, Lecture 1

65. Generalized intersection of convex sets, Proposition 9
66. Generalized union of convex sets, Proposition 10
67. Convex sets are intervals, Proposition 11
68. Open sets are countable unions of pairwise disjoint intervals, Theorem 2
69. Open covers, page 18
70. Compact sets, Definition 8.

Proposition 9 (From (65.)). If $\mathcal{C}$ is a collection of convex sets such that $\bigcap \mathcal{C} \neq \emptyset$, then $\bigcap \mathcal{C}$ is convex.

Proof. Let $x, y$ be two elements of $\bigcap \mathcal{C}$. Then $x, y \in C$ for every $C \in \bigcap \mathcal{C}$. Since $C$ is convex, $[x, y] \subseteq C$ for every $C \in \bigcap \mathcal{C}$, which implies $[x, y] \subseteq \bigcap \mathcal{C}$. \hfill \Box

Proposition 10 (From (66.)). If $\mathcal{D}$ is a collection of convex sets such that $\bigcap \mathcal{D} \neq \emptyset$, then $\bigcup \mathcal{D}$ is convex.

Proof. Let $x_0$ be an element of $\bigcap \mathcal{D}$. Let $x, y \in D := \bigcup \mathcal{D}$ be two elements. Then, there are $D_1$ and $D_2$ such that $x \in D_1$ and $y \in D_2$. Here we consider different cases:

- $x_0 \leq x \leq y$. Then $[x, y] \subseteq [x_0, y] \subseteq D_2 \subseteq D$ because $D_2$ is convex. If $x \leq y \leq x_0$, then $[x, y] \subseteq [x, x_0] \subseteq D_1 \subseteq D$ because $D_1$ is convex. Finally, if $x \leq x_0 \leq y$, we have $[x, x_0] \subseteq D_1$ and $[x_0, y] \subseteq D_2 \Rightarrow [x, y] \subseteq D_1 \cup D_2$. \hfill \Box

Proposition 11 (From (67.)). Convex sets are intervals.

Proof. Let $C$ be a convex non-empty set and let $x_0 \in C$ be an element of $C$. We define $C_n := C \cap (-n+x_0, n+x_0)$ for every natural number $n \geq 1$. We also define the collection of intervals

$$\mathcal{C} := \{C_n \mid n \geq 1\}, \quad C = \bigcup \mathcal{C}.$$  

From Proposition 9, $C_n$ is convex. Moreover, $\bigcap \mathcal{C} = (-1+x_0, 1+x_0) \neq \emptyset$. By Proposition 10, $\bigcup \mathcal{C}$ is convex. \hfill \Box

Theorem 2 (From (68.)). Let $\Omega$ be an open non-empty set of $\mathbb{R}$. Then $\Omega$ is countable union of open intervals which are disjoint from each other.

Proof. Let $x$ be an element of $\Omega$. Since $\Omega$ is open, there exists $r > 0$ such that $I_x(x) \subseteq \Omega$. We define the collection

$$\mathcal{A}_x := \{J \text{ open interval} \mid x \in J \subseteq \Omega\}.$$  

Clearly, $\mathcal{A}_x$ is non-empty, because $I_x(x) \in \mathcal{A}_x$. We also define $J_x := \bigcup \mathcal{A}_x$. The sets $J_x$ satisfy some properties that we list below

(i) $x \in J_x$
(ii) $J_x$ is an open set
(iii) $J_x$ is an interval
(iv) given $x, y \in \Omega$, either $J_x = J_y$ or $J_x \cap J_y = \emptyset$
(i). Since $I_r(x) \in \mathcal{A}$ and $x \in I_r(x)$, then $x \in J_x$ by definition of generalized union. 
(ii) $J_x$ is open because it is the union of a collection of open sets, namely $\mathcal{A}$. (iii) $J_x$ is an interval: $\mathcal{A}$ is a collection of intervals, that is, convex sets. Each of these sets contain $x$, from (i). Thus, $\cap \mathcal{A} \neq \emptyset$. From Proposition 10, $J_x$ is a convex set. From Proposition 11, $J_x$ is an interval.

(iv). Let us suppose that $J_x \cap J_y \neq \emptyset$. Then there exists $z$ in the intersection. Therefore, by Proposition 10 and Proposition 11, $J_x \cup J_y$ is an interval. It is open, because it is the union of two open sets. Since $x \in J_x \cup J_y$, there holds.

Then

$$J_x \cup J_y \in \mathcal{A}.$$ 

From (i), $\cup \mathcal{G} = \Omega$. We claim that $\mathcal{G}$ is countable. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $J \in \mathcal{G}$, the set $J \cap \mathbb{Q}$ is non-empty. We define the following function $f: \mathcal{G} \to \mathbb{Q}$, $f(J) = \phi(J \cap \mathbb{Q})$.

We define $w: \phi(J_1 \cap \mathbb{Q}) = \phi(J_2 \cap \mathbb{Q})$. Since $\phi$ is a choice function,

$$w \in (J_1 \cap \mathbb{Q}) \cap (J_2 \cap \mathbb{Q}) \subseteq J_1 \cap J_2.$$ 

From (iv), $J_1 = J_2$. \hfill \square

**Definition 8** (Compact sets, from (70.)) A non-empty set $E \subseteq \mathbb{R}$ is compact if for every open cover $\mathcal{U}$ there exists a finite sub-cover $\mathcal{U}' \subseteq \mathcal{U}$.

2016, March 24 - Week 4, Lecture 2

71. Solutions of the exercises of Week Three

Week 5, Lecture 1 - 2016, March 28

73. $\sigma$-algebras, Definition 9
74. $\sigma$-algebras generated by collections, Definition 10 and Proposition 12
75. the Borel’s $\sigma$-algebra, Definition, page 20
76. $G_\delta$ sets and $F_\sigma$ sets, page 20
77. properties of measures $m: \mathcal{B} \to [0, +\infty]$
77.1. translation invariance $m(A + y) = m(A)$
77.2. finite additivity, $m(A \cup B) = m(A) + m(B)$
77.3. $\sigma$-additivity: if $(A_n)$ is a disjoint countable collection

$$m(\bigcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} m(A_n)$$

77.4. $\sigma$-sub-additivity: if $(A_n)$ is a countable collection

$$m(\bigcup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} m(A_n)$$
77.5. $m^*(I) = \ell(I)$

78. length of intervals

78.1. bounded intervals: $I \in \{(a, b), [a, b), (a, b], [a, b]\}$, $\ell(I) := b - a$

78.2. unbounded intervals: $\ell(I) = \infty$

79. length of collection of intervals, Definition 11

80. the Lebesgue outer measure, §2.2, page 31.

Definition 9 ($\sigma$-algebras, from (73.)). A collection of sets $\mathcal{B}$ is a $\sigma$-algebra if it satisfies the following properties:

(i) $\emptyset \in \mathcal{B}$
(ii) $E \in \mathcal{B} \Rightarrow E^c \in \mathcal{B}$
(iii) given a countable collection $G \subseteq \mathcal{B}$, there holds $\bigcup G \in \mathcal{B}$.

Definition 10 (Length of collection, from (74.)). Given a collection $A \subseteq \mathcal{P}(\mathbb{R})$, we define

$$F_A = \{ B \mid (A \subseteq B) \land (B \text{ is a } \sigma\text{-algebra}) \subseteq \mathcal{P}(\mathcal{P}(\mathbb{R})) \}.$$ 

We define $B(A) := \cap F_A$.

Proposition 12 (From (74.).) $B(A)$ is a $\sigma$-algebra.

Proof. 

(i) For all $B$ we have $\emptyset \in B \in F_A$. Then $\emptyset \in B(A)$
(ii) $E \in B(A)$ implies $E \in B \in F_A$ for every $B$. Then $E^c \in B$, hence $E^c \in F_A$
(iii) let $G \subseteq B(A)$ be a countable collection. Then, $G \subseteq B$ for every $B \in F_A$. Then $B \subseteq \cap F_A$.

□

Definition 11 (Length of collection, from (79.)). Given a countable collection of intervals $J$, we define its length as $L(J) := \sum_{n=1}^{+\infty} \ell(I_n)$, where $I_n \in J$.

Week 10, Lecture 1 - 2016, May 2

81. Monotonicity of the outer measure, page 31

82. the outer-measure of a countable set is zero, Example in page 31

83. given two bounded intervals $I_1, I_2$, $\ell(I_1 \cup I_2) \leq \ell(I_1) + \ell(I_2)$

84. Proposition 13

85. the outer measure of an interval is equal to its length, Theorem 3.

Proposition 13 (From (84.).) If $[a, b] \subseteq \cup J$ there $J$ is a disjoint collection of open interval, then there exists $I \in J$ such that $[a, b] \subseteq I$.

Proof. Suppose that there are two intervals $I_1 \neq I_2$ such that $a \in I_1$ and $b \in I_2$. Without loss of generality we can suppose that $I_1$ is bounded from above, $I_2$ is bounded from below and $\sup(I_1) = S = \inf(I_2)$. Since all the intervals are open, $s \notin \cup J$. Since $a \in I_1$, we have $a < s$. Since $b \in I_2$, we also have $s < b$. Therefore, $s \in [a, b] \subseteq \cup J$ gives a contradiction. □

Theorem 3 (From (85.).) If $A$ is an interval, then $m^*(A) = \ell(A)$.

Proof. Firstly, we consider bounded intervals and, in particular, the closed bounded interval $[a, b]$. For every natural number $n$, we have the collection

$$J_n := \{(a - 1/2n, b + 1/2n)\}, \quad [a, b] \subseteq \cup J_n.$$
Then \( m^*([a, b]) \leq L(J_n) = b - a + 1/n \). The last inequality holds for every \( n \in \mathbb{N} \). Then \( m^*([a, b]) \leq b - a \).

We prove the converse inequality. Given \( n \in \mathbb{N} \), there exists \( J_n \) such that \([a, b] \subseteq J_n \) and \( L(J_n) \leq m^*(A) + 1/n \). By the Heine-Borel’s theorem, there exists a finite subcover \( J_n' \). We claim that \( L(J_n') \geq b - a \) and prove the claim by induction on \( k := \#J_n' \). If \( k = 1 \), then \( J_n' \) contains exactly one interval, namely, \( I = (c, d) \). Since \([a, b] \subseteq (c, d) \) we have \( c < a < b < d \).

Then \( L(J_n') = \ell(I) = d - c > b - c > b - a = \ell([a, b]) \). Now, we prove that \( k \Rightarrow k + 1 \).

We consider the two different cases:

First case. \( J_n' \) is a disjoint collection of open intervals. Then, by Proposition 13, there exists \( I \in J_n' \) such that \([a, b] \subseteq I \). Then

\[
\ell([a, b]) \leq \ell(I) = L(J_n') \geq L(J_n) \geq m_*(([a, b]) + 1/n).
\]

Taking the limit, we obtain \( b - a \leq m_*(([a, b])) \).

Second case. There are two intervals \( I', I'' \in J_n' \) such that \( I' \neq I'' \) and \( I' \cap I'' \neq \emptyset \). Then we define \( \bar{I} := I' \cup I'' \) which is interval because the intersection is non-empty. We define

\[
J_n'' := J_n' \cup \{\bar{I} \} - \{I', I''\}
\]

which is an open cover of \([a, b]\) and \( \#J_n'' = k \). Then, by the inductive hypothesis,

\[
b - a \leq L(J_n'') \leq L(J_n') \]

The second inequality follows from \( \ell(\bar{I}) \leq \ell(I') + \ell(I'') \). This settles the second case. Finally,

\[
b - a \leq L(J_n') \leq L(J_n) \leq m_*(([a, b]) + 1/n).
\]

Taking the limit, we obtain \( b - a \leq m_*(([a, b])) \).

Other bounded intervals. Given \( n \geq 1 \), we consider the set \([a + 1/2n, b - 1/2n]\). Then \((a, b) \supseteq [a + 1/2n, b - 1/2n]\). Since the outer measure is monotone, from the inclusions

\[
[a, b] \supseteq [a, b], (a, b) \supseteq (a, b) \supseteq [a + 1/2n, b - 1/2n].
\]

We obtain

\[
m^*([a, b]) \geq m^*([a, b]), m^*((a, b)) \geq m^*([a, b]) \geq m^*([a + 1/2n, b - 1/2n])
\]

then

\[
b - a \geq m^*([a, b]), m^*((a, b)) \geq m^*([a, b]) \geq b - a - 1/n.
\]

Taking the limit, we obtain

\[
b - a = m^*([a, b]) = m^*((a, b)) = m^*([a, b])
\]

which is equal to the length of each of those intervals.

Unbounded intervals. We use the monotonicity property of the outer-measure. From the inclusions

\[
[a, +\infty) \supseteq (a, +\infty) \supseteq (a, n]
\]

we obtain that \( m^*((a, +\infty)) \geq n - a \) for every \( n \in \mathbb{N} \). Then \( m^*([a, +\infty)) = m^*((a, +\infty)) = \infty \).

\[
(-\infty, b] \supseteq (-\infty, b] \supseteq (-n, b].
\]

Then \( m^*((-\infty, b]) \geq b + n \). Then \( m^*([-\infty, b]) = m^*([-\infty, b]) = \infty \). Finally,

\[
(\forall n \in \mathbb{N}) : \ R \supseteq (-\infty/2, n/2) \Rightarrow m^*(R) \geq n
\]

which implies that \( m^*(\mathbb{R}) = \infty \). □