SOLUTIONS OF EXERCISES WEEK THREE

Exercise 1. The statement $P \lor Q$ correspond to "$P$ or $Q$".
Using the symbols $\land$, $\lor$ how can we express "$P$ or $Q$, but only one of them".

Solution. $(P \lor Q) \land \neg (P \land Q)$.

Exercise 2. Suppose that we are dealing only with sets (no proper classes) and that the Class Construction Axiom of Cantor holds. During the lectures, we defined $A \in T \iff A$ is an infinite set and showed that $T \in T$. Prove that $T - \{T\} \in T - \{T\}$.

Solution. We set $S := T - \{T\}$. Since $T$ is infinite, $S$ is infinite. Then $S \in T$. Since $T \in T$, we also have $T - \{T\} \subseteq T$. Therefore, $S \neq T$. Then $S \in T - \{T\}$.

Exercise 3. Let $f : A \to B$ be a function. Given two subclasses $C_1, C_2 \subseteq A$, show that

$$\bar{f}(C_1 \cup C_2) = \bar{f}(C_1) \cup \bar{f}(C_2),$$

$$\bar{f}(C_1 \cap C_2) \subseteq \bar{f}(C_1) \cap \bar{f}(C_2).$$

Solution. Given an element $y$, we have

$$y \in \bar{f}(C_1 \cup C_2) \iff \exists x \in C_1 \cup C_2 \text{ s.t. } f(x) = y$$
$$\iff (\exists x \in C_1 \text{ s.t. } f(x) = y) \lor (\exists x \in C_2 \text{ s.t. } f(x) = y)$$
$$\iff (y \in \bar{f}(C_1)) \lor (y \in \bar{f}(C_2)).$$

Given an element $y$, we have

$$y \in \bar{f}(C_1 \cap C_2) \Rightarrow \exists x \in C_1 \cap C_2 \text{ s.t. } f(x) = y$$
$$\Rightarrow (\exists x \in C_1 \text{ s.t. } f(x) = y) \land (\exists x \in C_2 \text{ s.t. } f(x) = y)$$
$$\Rightarrow (y \in \bar{f}(C_1)) \land (y \in \bar{f}(C_2)).$$

Exercise 4. Let $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = 2^n - 1$. Is the function injective, surjective or bijective?

Solution. $f$ is injective, as $2^n = 2^m \Rightarrow n = m$. $f$ is not surjective because $2^n - 1$ is always an odd number. Then, for instance $2 \notin \text{ran}(f)$.

Exercise 5. Is $\mathbb{N}$ equipotent to $\mathbb{N} - \{1\}$? Is $\mathbb{N}$ equipotent to $\mathbb{N} \cup \{\pi\}$? if the answer is yes, find a bijective function.

Solution. $\mathbb{N} \approx \mathbb{N} - \{1\}$. In fact $g(n) = n + 1$ is a bijective function from $\mathbb{N} - \{1\}$ to $\mathbb{N}$. While

$$h : \mathbb{N} \cup \{\pi\} \to \mathbb{N}, \quad h(x) = \begin{cases} 1 & \text{if } x = \pi \\ x + 1 & \text{if } x \neq \pi \end{cases}.$$
This function is surjective. In fact, given \( n \in \mathbb{N} \), if \( n = 1 \), then \( n = h(n) \), so \( n \in \text{ran}(h) \). If \( n \geq 2 \), then \( n = h(n + 1) \). It is injective, because if \( x, y \neq \pi \) we have
\[
h(x) = h(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y.
\]

If \( x = \pi \) and \( y \neq \pi \), we have
\[
h(x) = h(y) \Rightarrow 1 = y + 1 \Rightarrow y = 0
\]
but, by definition of \( \mathbb{N} \), the zero is not an element of \( \mathbb{N} \).

Exercise 6. Let \( f \) be the function defined from \( A = \{0, 1, 2\} \) to \( B = \{3, 4\} \) as
\[
f(0) = 3, \ f(1) = 3, \ f(2) = 4.
\]
Is \( f \) injective or surjective? Is there a function \( g: B \rightarrow A \) such that \( g \circ f = \text{id}_A \)? How many of them?

Solution. \( f \) is surjective. In fact, \( B \subseteq \text{ran}(f) \) as \( 3 = f(0) \) and \( 4 = f(1) \). The function is not injective because \( f(0) = f(1) = 3 \). There is no such function \( g \) because, on the contrary,
\[
g \circ f(x) = x
\]
for every \( x \in A \) would imply
\[
g(f(0)) = 0, \quad g(f(3)) = 3.
\]
However, \( f(0) = f(1) = 3 \). Then \( g(3) = 0 = 3 \) which gives a contradiction. Then, there are no functions \( g \) such that \( g \circ f = \text{id}_A \).

Exercise 7. Let \( A \) and \( B \) be two classes such that \( \#A = n \) and \( \#B = m \). How many different functions are \( f: A \rightarrow B \).

Exercise 8. For each element of \( A \) we have \( m \) different choices in \( B \). Then, there are \( m^n \) functions.

Exercise 9. In the example

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What are the sets and proper classes? which of the following classes exist
\[
x \cap y, \ y \cap z, \ x \cup z, \ \{x\}, \ \{y\}, \ \{D\}, \ \emptyset, \ \mathcal{U}?
\]
Is Axiom 2 satisfied? is Axiom 3 satisfied?

Solution.
Sets: \( x, y, z \).
Proper classes: \( D \).
\( x \cap y = \{x\} \cap \emptyset = \emptyset = y. \) So, \( x \cap y \) exists because \( x \cap y = y. \)
\( y \cap z = y. \)
\( x \cup z = \{x\} \cup \{y, z\} = \{x, y, z\} \) which exists because it is equal to \( D. \)
\( \{x\} = x. \)
\( \emptyset \{y\}. \)
\#\{D\} because \(D\) is a proper class.
\[\emptyset = y.\]
\[\mathcal{U} = D.\]
Axiom 2 is not satisfied, because, for example \(\{y\}\) does not exist. In order to check Axiom 3, we look at the pairs
Pairs: \(x, z\).
We compare the pairs with the sets, which are \(x, y, z\). Then all the pairs are sets. Then Axiom 3 holds. \(\square\)
SOLUTIONS OF THE EXERCISES WEEK FIVE

Exercise 1. Let \( f : A \rightarrow B \) be a function such that there are two functions \( g_1, g_2 : B \rightarrow A \) such that
\[
g_i \circ f = id_A, \quad f \circ g_i = id_B.
\]
Prove that \( g_1 = g_2 \).

Solution. We take the composite function
\[
(g_1 \circ f) \circ g_2 = id_A \circ g_2 = g_2.
\]
On the other hand
\[
(g_1 \circ f) \circ g_2 = g_1 \circ (f \circ g_2) = g_1 \circ id_B = g_1.
\]
Then \( g_1 = g_2 \).  \( \square \)

Exercise 2. Given three classes \( A, B \) and \( x \), is it true or false that
(a) \( A \subseteq B \Rightarrow A - x \subseteq B - x \)
(b) \( A \in B \Rightarrow A - x \in B - x \).

Solution. (a). It is true. If \( y \in A - x \), then \((y \in A) \land (y \notin x)\). Since \( A \subseteq B \), we have
\[
(y \in A) \land (y \notin x) \Rightarrow (y \in B) \land (y \notin x).
\]
(b). It is false. For instance, \( A = \{0,1\} \) and \( B = \{\{0,1\}\} \), clearly, \( A \in B \). If \( x = \{0\} \), then
\[
A - x = \{1\} \notin B - x = B.
\]
\( \square \)

Exercise 3. Let \( x := (a, b) \) be an ordered pair. Find the following classes:
\[
\cup (\cap x), \quad \cap (\cup x - \cap x).
\]

Solution. We have
\[
\cap x = \{a, b\} \cap \{a\} = \{a\}.
\]
Then \( \cup (\cap x) = a \). If \( a \neq b \), then
\[
\cup x - \cap x = \{b\}, \quad \cap (\cup x - \cap x) = b.
\]
If \( a = b \), then \( \cup x - \cap x = \emptyset \), so the generalized intersection is not defined.  \( \square \)

Exercise 4 (A2 + A3 + A4). Suppose that the three axioms A2, A3 and A4 hold. Suppose that there exists a set \( x \). Then, there are infinitely many sets.

Solution. From A2, there exists \( \emptyset \). From A4, \( x \subseteq \emptyset \) implies \( \emptyset \) is a set. Now, suppose that there are finitely many sets many sets
(1) \( \emptyset, x_2, \ldots, x_n \)
all different from each other. Still, from A3, we have the sets
(2) \( \emptyset, \{x_2\}, \{x_3\}, \ldots, \{x_n\} \)
all different from each other. Moreover, there is the emptyset. So, there are \( n + 1 \) sets.  \( \square \)

Date: 2016, April 11.
**Exercise 5.** Let $A$ and $B$ be two classes, both non-empty. Are the following implications true or false?

(a) $A \subseteq B \Rightarrow \cup A \subseteq \cup B$
(b) $A \subseteq B \Rightarrow \cap B \subseteq \cap A$.

Are the converse implications true?

**Solution.**
(a). $x \in \cup A \Rightarrow \exists y \in A \text{ s.t. } x \in y$. Since $A \subseteq B$, we have 
\[ \exists y \in A \text{ s.t. } x \in y \Rightarrow \exists y \in B \text{ s.t. } x \in y \Rightarrow x \in \cup B. \]
The converse is not true. For instance, if $a, b, c$ are three sets different from each other, we define 
\[ A = \{\{a, b\}, \{b, c\}\}, \quad B = \{\{a, b, c\}\}, \quad A \not\subseteq B. \]
However, 
\[ \cup A = \cup B = \{a, b, c\}. \]
(b). $x \in \cap B \Rightarrow (\forall y \in B) x \in y$. Since $A \subseteq B$, we have $x \in y$ for every $y \in A$. Then $x \in \cap A$. The converse implication is not true. Consider the example 
\[ A = \{\{a\}\}, \quad B = \{\{b\}, \{c\}\}. \]
Then $A \not\subseteq B$, but $\cap B = \emptyset \subseteq \cap A$. □

**Exercise 6.** Is $A^2$ equivalent to any of the following statements?

(a) given a statement $p(x)$ there exists a class $R$ such that $p(R)$ it is true
(b) given a statement $p(x)$ there exists a class $R$ such that $\ x \in R \iff x$ is a set and $x \in R$.
(c) given a statement $p(x)$ there exists a set $R$ such that $\ x \in R \iff x$ is a set and $p(x)$ is true
(d) given a statement $p(x)$ there exists a class $R$ such that $\ x \in R \iff p(x)$ is true.

**Solution.**
(a). $A^2$ does not imply (a). In fact, here "$p(x)$" has been replaced with "$p(R)$". For instance, if we define $p(x) : x \not\in x$, there is not class $R$ satisfying the property $p$.
(b). (b) does not imply $A^2$. In the statement "$p(x)$" does not appear at all. Consequently, the new statement, "there exists a class $R$ such that $x \in R \iff x$ is a set and $x \in R$" is always true, regardless whether $A^2$ is satisfied or not.
(c). $A^2$ is equivalent to (c).
(d). $A^2$ does not imply (d). An example is given from the Russell Class. □

**Exercise 7.** Is it true that $R \approx R - \{0\}$? is yes, find a bijective function.

**Solution.** We can define explicitly a bijective function 
\[ g(x) := \begin{cases} 
  x + 1 & \text{if } x \in \mathbb{N} \cup \{0\} \\
  x & \text{if } x \not\in \mathbb{N} \cup \{0\}. 
\end{cases} \]
Exercise 8. Let \( f : A \to B \) be a function. Given two subclasses \( D_1, D_2 \subseteq A \), show that
\[
\tilde{f}(D_1 \cup D_2) = \tilde{f}(D_1) \cup \tilde{f}(D_2)
\]
\[
\tilde{f}(D_1 \cap D_2) = \tilde{f}(D_1) \cap \tilde{f}(D_2).
\]

Solution.
\[
x \in \tilde{f}(D_1 \cup D_2) \iff f(x) \in D_1 \cup D_2 \iff (f(x) \in D_1) \lor (f(x) \in D_2)
\]
\[
\iff (x \in \tilde{f}(D_1)) \lor (x \in \tilde{f}(D_2)).
\]
\[
x \in \tilde{f}(D_1 \cap D_2) \iff f(x) \in D_1 \cap D_2 \iff (f(x) \in D_1) \land (f(x) \in D_2)
\]
\[
\iff (x \in \tilde{f}(D_1)) \land (x \in \tilde{f}(D_2)).
\]
\( \Box \)
EXERCISES WEEK SEVEN

Exercise 1 (A1 + A2 + A3). Given \(a, b\) be two sets. Prove that \(\bigcup (A \times B) = A \cup B\).

Exercise 2 (A2 + A3). Let \(f : A \to B\) be a function. As a function \(f\) is also a class. Prove that \(f \approx A\).

Exercise 3. Prove that \(N \approx N \times \{0,1\}\) (that is, find a bijective function).

Exercise 4. In the set of natural numbers, consider the order relation given by \((n, m) \in R\) if and only if \(n \mid m\). Find a maximal chain.

Exercise 5 (A1 + A2 + A3 + A4). Let \(A, B\) be two classes. Prove that there are two classes \(C \approx A\) and \(D \approx B\) such that \(C \cap D = \emptyset\).
**EXERCISES OF WEEK NINE**

**Exercise 1.** Let $C_1, C_2 \subseteq A$ be two chains in the order relation $(A, \leq)$. Is $C_1 \cap C_2$ a chain?

**Exercise 2.** In $(A, \leq)$ we consider the relation

$$xGy : x \text{ is comparable to } y.$$ 

Prove that if $(A, \leq)$ is a FOC, then $G$ is an equivalence relation. Is the converse true? that is, if $G$ is an equivalence relation is it true that $(A, \leq)$ a FOC?

**Exercise 3.** Let $A$ be a non-empty class. Prove that every Choice Function is surjective.

**Exercise 4.** During the final exam, a student has to solve the following exercise:

"find a Choice Function for the set $A = \{0, 1, 2, 3\}$".

Unfortunately, he did not study Choice Functions. The only thing he remembers is that a Choice Function is a function defined on $\mathcal{P}(A)^*$ to $A$, but nothing else. Then, he decides to write on his exam sheet a randomly chosen function from $\mathcal{P}(A)^*$ to $A$, hoping that it will match a Choice Function.

(1). What are the chances that he will give a correct answer to the exercise?

(2). By the way, find a Choice Function for the set $A = \{0, 1, 2, 3\}$.

**Exercise 5.** In a homework for a course of Set Theory, a professor gives to the students an exercise where they have to find an injective Choice Function on a particular set given by the professor. All the students solve the exercise and find the same Choice Function! However, they worked independently on the homeworks and never met or send messages to each other during the weekends.

Can you explain how this is possible?

**Exercise 6 (A1-A6).** Let $A$ and $B$ be two non-empty sets. We define $S$, as the class of the functions $f$ such that $\text{dom}(f) \subseteq A$ and $\text{ran}(f) \subseteq B$. Prove that $S$ is a set.