**EXERCISES FOR ORAL EXAMS**

**Exercise 1.** Is the exponential map \( \exp : \mathbb{C} \rightarrow \mathbb{C} \) injective or surjective? Show that it is locally invertible.

**Exercise 2.** In the linear space of polynomials of degree \( \leq n \), \( \mathbb{R}_n[X] \), you can define the linear map

\[
T : p \mapsto e^{-X} \int e^t p(t) dt.
\]

Prove that \( T(\mathbb{R}_n[X]) \subseteq \mathbb{R}_n[X] \) and that it is invertible. \(^1\)

**Exercise 3.** Let \( f_1 \) and \( f_2 \) two linear application on a space \( V \). Find necessary and sufficient conditions which ensure the existence of a linear application \( g \neq 0 \) such that

\[
g \circ f_1 = f_1 \circ g = g \circ f_2 = f_2 \circ g = 0
\]

**Exercise 4.** Let \( V \) be an \( \mathbb{R} \)-linear space of finite dimension. Prove that the set \( \text{GL}(V) \) of linear invertible maps generates the linear space of linear maps \( \mathcal{L}(V) \). \(^2\)

**Exercise 5.** Give an example of linear space \( X \) and function \( f : X \rightarrow X \) such that the sequences \( \ker(f^i) \) and \( \text{Img}(f^i) \) are not stable.

**Exercise 6.** In \( \mathbb{R}[X] \), define the set \( S_n = \{ p \in \mathbb{R}[X] \mid \#Z_p = n \} \), where \( Z_p \) is the zeroes set of \( p \). Is it true that \( \text{Span}(S_n) = \mathbb{R}[X] \)?

**Exercise 7.** Let \( N \) be a subspace of the space of linear maps \( \mathcal{L}(V) \), where \( V \) is a finite-dimensional linear space. Suppose that in every pair of elements of \( N \), two maps commute with each other, and every element in \( N \) is nilpotent. Prove that there exists \( v \neq 0 \) such that \( f(v) = 0 \) for every \( f \in N \). \(^3\)

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\(^1\)This exercises was taken from the textbook "Problemi Scelti di Analisis Matematica I" authored by E. Acerbi, L. Modica and S. Spagnolo

\(^2\)This exercises was taken from the final exam of the course of "Geometria I" of R. Benedetti, M. Ferrarotti and E. Fortuna on June 1997

\(^3\)This exercise is a preliminary Lemma to the Engels’ Theorem, P. Humpreys, "Introduction to Lie Algebras and Representation Theory"