On the Orbital Stability of Standing-Wave Solutions to a Coupled Non-Linear Klein-Gordon Equation

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Abstract
We show the existence of standing-wave solutions to a coupled non-linear Klein-Gordon equation. Our solutions are obtained as minimizers of the energy under a two-charges constraint. We prove that the ground state is stable and that standing-waves are orbitally stable under a non-degeneracy assumption.

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1 Introduction
This work is on the orbital stability of standing-wave solutions
\[ v_j(t, x) = e^{-i\omega_j t} \psi_j(x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad 1 \leq j \leq 2 \] (1.1)
to the coupled non-linear Klein-Gordon equation
\[ \Box v_j + m_j^2 v_j + \partial_{\psi_j} F(v) = 0, \quad 1 \leq j \leq 2. \] (CNLKG)
The m_j’s are positive real numbers and
\[ F : \mathbb{C}^2 \to \mathbb{C} \]

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A subset is stable if, for every ε > 0, there exists δ > 0 such that
d(Φ, S) < δ ⇒ d(U(t, Φ), S) < ε
for every t ≥ 0.

Given (z, w) ∈ C² × C², we define

\[(z \cdot w)_j := z_j w_j,\]

Following this notation, if v is a standing-wave as in (1.1), then

\[(v(0, ·), \partial_t v(0, ·)) = (u, -iu \cdot u).\]
Definition 1.2 A standing-wave is orbitally stable if the subset of $X$

$$\Gamma(u, \omega) = \left\{ (\lambda \cdot u(\cdot + y), -i\omega \cdot \lambda \cdot u(\cdot + y)) | (\lambda, y) \in \mathbb{T}^2 \times \mathbb{R}^N \right\}$$

is stable.

From (A3), if $v$ is a standing-wave solution to (CNLKG), then $(u, \omega)$ is a solution to the elliptic system

$$-\Delta u_j + m_j^2 u_j + \partial_j F(u) = \omega_j^2 u_j, \quad 1 \leq j \leq 2. \quad (1.3)$$

In order to solve (1.3), we follow the variational approach of [3], where the energy functional and the constraint are provided by conserved quantities: we refer to

$$E \ni \left( u \subset \mathbb{R}^N, \omega \in \mathbb{R}^2 \right) \mapsto \mathcal{E}(u, \omega) := \frac{1}{2} \sum_{j=1}^2 \left( \int_{\mathbb{R}^N} |\phi_j'|^2 + |D\phi_j|^2 + 2V(\phi) \right)$$

and

$$X \ni (\phi, \phi_t) \mapsto \mathcal{C}_j(\phi, \phi_t) := -\operatorname{Im} \int_{\mathbb{R}^N} \phi_j^* \partial_j \phi_j, \quad 1 \leq j \leq 2$$

as energy and charges. By (A3), the functions

$$\mathbb{R} \ni t \mapsto e(t) := \mathcal{E}(v(t, \cdot), \partial_x v(t, \cdot)), \quad \mathbb{R} \ni t \mapsto c_j(t) := \mathcal{C}_j(v(t, \cdot), \partial_x v(t, \cdot)) \quad (1.4, 1.5)$$

are constant for every solution $v$. In particular, if $v$ is a standing-wave as in (1.1), then

$$e(0) = \mathcal{E}(u, -i\omega \cdot u) = \frac{1}{2} \sum_{j=1}^2 \int_{\mathbb{R}^N} \left( |Du_j|^2 + m_j^2 u_j^2 + \omega_j^2 u_j^2 \right) + \int_{\mathbb{R}^N} F(u). \quad (1.6)$$

and

$$c_j(0) = \mathcal{C}_j(u, -i\omega \cdot u) = \omega_j \int_{\mathbb{R}^N} |u_j|^2. \quad (1.7)$$

We define the energy functional

$$E : H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R}, \quad E(u, \omega) := \mathcal{E}(u, -i\omega \cdot u)$$

and the constraint

$$C_j : H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R}, \quad C_j(u, \omega) := \omega_j \int_{\mathbb{R}^N} |u_j|^2$$

$$M_C := \{(u, \omega) | C_j(u, \omega) = C_j \}.$$  

The key observation made in [3, Theorem 2.6] is that critical points of $E$ constrained to $M_C$ are classic solutions to (1.3). In Proposition 2.2 we prove this fact for the coupled case and that each of the components $u_j$ does not change sign.

The main theorems of this work are the following:

Theorem 1.1 Given a minimising sequence $(u_n, \omega_n)_{n \geq 1}$ for $E$ over $M_C$, there exists a minimiser $(u, \omega)$ and $(y_n)_{n \geq 1} \subset \mathbb{R}^N$ such that, up to extract a subsequence,

$$(u_n, \omega_n) = (u(\cdot + y_n), \omega) + o(1).$$
The proof is carried out by proving a concentration behaviour of the minimising sequences of the functional
\[ J(u) = \frac{1}{2} \sum_{j=1}^{N} \int_{\mathbb{R}^N} |Du_j|^2 + \int_{\mathbb{R}^N} F(u) \]
on the constraint
\[ N_\rho := \{ u \ | \ ||u||_{L^2(\mathbb{R}^N)} = \rho \}. \]
In turn, such behaviour follows from the sub-additivity property of the function \( I(\rho) := \inf_{N_\rho} J \)
\[ I(\rho) < I(\tau) + I(\rho - \tau), \quad 0 < \tau_j \leq \rho_j, \tau \neq \rho. \tag{1.8} \]
Such property plays a crucial role in the proof of the orbital stability of standing-wave solutions to a variety of evolution problems: the non-linear Schrödinger equation, \([11, 4]\), coupled NLS in dimension \( N = 1 \), in \([21]\), and KdV-NLS systems \([1]\). In these references, (1.8) is obtained through rescaling argument (as in \([4]\)) or symmetries arising from the choice of the non-linear term (as in \([21]\)). Due to the lack of suitable rescaling arguments for non-linearities satisfying \((A_1)\), we obtain (1.8) from considerations on the gradient terms. We exploit an idea carried out by J. Byeon in \([10, \text{Proposition 1.4}]\) which is based on the symmetric rearrangement and we prove that, if
\[ (u, v) \in N_\tau \times N_{\rho - \tau} \]
have disjoint support and are a good approximation of \( I(\tau) \) and \( I(\rho - \tau) \), respectively, then there exists \( D = D(\rho, \tau) > 0 \) such that
\[ \| Dw^* \| \leq \| Du \| + \| Dv \| - D, \]
where \( w = u + v \) and \( w^* \) is the Steiner symmetrization of \( w \). We prove this fact in Lemma 3.1.

Preliminary notation is required to introduce the next results. To \( C \in \mathbb{R}^2 \), we can associate the subset
\[ m_C := \inf_{M_C} E, \quad K_C := \{ (u, \omega) \in M_C \ | \ E(u, \omega) = m_C \} \]
and
\[ \Gamma_C = \bigcup_{(u, \omega) \in K_C} \Gamma(u, \omega). \]

\textbf{Theorem 1.2} Let \( C \in \mathbb{R}^2 \) be such that \( C_j \neq 0 \) for \( j = 1, 2 \). Given a sequence
\[ (\Phi_n)_{n \geq 1} \subset X \]
then \( d(\Phi_n, \Gamma_C) \to 0 \) if and only if
\[ E(\Phi_n) \to m_C \text{ and } C_j(\Phi_n) \to C_j. \]
A proof of this theorem in the scalar case can be found in \([3, \text{§3.1}]\). We included a proof which does not use the local well-posedness of (CNLKG) (implicitly used in \([3, \text{Lemma 3.5}]\)). Our proof relies on an improved version of the Convexity Inequality for Gradients, \([18, \text{Theorem 7.8, p., 177}]\), outlined in Lemma 5.1. Theorem 1.2 implies that
\[ X \ni \Phi \mapsto V(\Phi) := (E(\Phi) - m_C)^2 + \sum_{j=1}^{2} (C_j(\Phi) - C_j)^2 \tag{1.9} \]
is a Lyapunov function for \( \Gamma_C \) (see \([3, \text{Definition 2.4}]\)).
Given a subset $S \subset H^1 \times \mathbb{R}^2$ and $(u, \omega)$ in $K_C$, we define the following subsets of $H^1 \times \mathbb{R}^2$:

$$B_\delta(S) := \{(w, \alpha) | d((w, \alpha), S) < \delta\}, \quad G(u, \omega) := \{(u(\cdot + y), \omega) | y \in \mathbb{R}^N\}.$$ 

We say that $(u, \omega)$ satisfies the condition (D) if there exists $\delta > 0$ such that, for every $(u', \omega') \in K_C \setminus \{(u, \omega)\}$ such that

$$\Gamma(u', \omega') \not\sim \Gamma(u, \omega),$$

there holds

$$B_\delta(G(u, \omega)) \cap G(u', \omega') = \emptyset. \quad (D)$$

**Theorem 1.3** The subset $\Gamma_C \subset X$ is stable. For every minimiser $(u, \omega)$ fulfilling condition (D), $\Gamma(u, \omega)$ is stable.

We intentionally restricted our work to the higher dimensional case $N \geq 3$ and to $C_1 C_2 \neq 0$. We address to further works the treatment of the semi-trivial case ($C_j = 0$ for some $1 \leq j \leq 2$), and the lower dimensions $N = 1, 2$.

Results on the orbital stability of standing-wave solutions to coupled non-linear Klein-Gordon equations, with a different variational characterisation, have been obtained in [30]. Numerical results on the existence of coupled standing-waves have been obtained in [8] when $N = 3$ and the non-linearity has a critical exponent.

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## 2 Basic properties of the functional $J$

**Proposition 2.1** For every $p$ in $\mathbb{R}^2$ with $\rho_j > 0$,

(i) $J$ attains negative values on $N_p$;

(ii) $J$ is bounded from below and minimising sequences of $J$ over $N_p$ are bounded; moreover,

(iii) $J$ is continuous;

(iv) given a weakly converging sequence $u_n \rightharpoonup u$, up to extract a subsequence

$$J(u_n - u) = J(u_n) - J(u) + o(1).$$

**Proof.** (i) By choosing a test function in a neighbourhood of the origin we can write $F = F_0 + F_\infty$, where

$$|F_0(z)| \leq c|z|^p, \quad |F_\infty(z)| \leq c|z|^q.$$ 

A sequence $(u_n)_{n \geq 1}$ such that $u_n \rightharpoonup u$ in $H^1$, converges in $L^p(\mathbb{R}^N)$ by the Sobolev inequality

$$\|u\|_{L^p} \leq S \|u\|_{L^2}^{1 - \frac{N}{2} + \frac{N}{p}} \|Du\|_{L^2}^{\frac{N}{2} - \frac{N}{p}}.$$

(3.1)
There exists \( g \) in \( L^p(\mathbb{R}^N) \) and a subsequence \((u_{n_k})_{k \geq 1}\) such that
\[
|u'_{n_k}| \leq g
\]
and \( u_{n_k} \to u \) pointwise a.e., by [6, Théorème IV.9, p. 58]. Then
\[
\int_{\mathbb{R}^N} F_0(u_{n_k}) \to \int_{\mathbb{R}^N} F_0(u)
\]
by the dominated convergence theorem. We can extract a subsequence alike from every subsequence of \((u_{n_k})_{n \geq 1}\). Then, the map \( u \mapsto \int F_0 \circ u \) is continuous, and the gradient part of \( J \) is smooth. An adaptation of the technique used in [2, Theorem 2.6, p. 17] would allow to conclude that \( J \) is \( C^1(H^1, \mathbb{R}) \).

(ii) We refer to Step II of the Appendix of [4], which addresses the scalar case.

(iii) Following the proof of [4, Lemma 5], we can show that \( J \) attains negative values on \( N_{\rho} \) for every choice of \( \rho \): setting
\[
\lambda := (\rho_1^{-1} \rho_2)^{1/2}
\]
and
\[
u := (w, \lambda w), \ w \in N_{\rho_1},
\]
we have
\[
J(u) = (1 + \lambda^2)^{-1} J_1(w),
\]
where
\[
J_1(w) := \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 + \int_{\mathbb{R}^N} F_1(w)
\]
\[
F_1(s) := (1 + \lambda^2)^{-1} \left( -\mu \lambda |s|^{2\gamma} + G(s, \lambda s) \right).
\]
By (A_1) and (A_2) the non-linearity \( F_1 \) fulfils hypotheses \( (F_1) \) and \( (F_2) \) of [4]. Then, by [4, Lemma 5], there exists \( w \) such that \( J_1(w) < 0 \). Then \( J(u) < 0 \).

(iv) By the Hölder inequality and (A_3), we have
\[
J(u) \geq \frac{1}{2} \sum_{j=1}^{2} \|Du_j\|_{L^2}^2 - 2\mu \sum_{j=1}^{2} \|u_j\|_{L^2} \|u_{j'}\|_{L^2}^{\gamma'}
\]
\[
\geq \frac{1}{2} \sum_{j=1}^{2} \left( \|Du_j\|_{L^2}^2 - \mu \|u_j\|_{L^2} \|u_{j'}\|_{L^2}^{\gamma'} \right),
\]
From (3.1), there exists a constant \( c' \) such that
\[
J(u) \geq c' \sum_{j=1}^{2} \|Du_j\|_{L^2}^2 - \|Du_j\|_{L^2}^{2\left(\frac{2}{p'} - 1\right)}.
\]
By the hypotheses on \( \gamma \) in (A_3), the right member of the above inequality is bounded from below, as \( J \) is. Given a minimising sequence, \((u_{n_k})_{n \geq 1}\) in \( N_{\rho} \), for \( n \) large we have \( J(u_{n_k}) < 0 \), by (i). Then, \( \|Du_{n_k}\|_{L^2} \) is bounded by (3.3). Because \( \|u'_{n_k}\|_{L^2}^2 \equiv \rho_j \), the \( H^1 \)-norm is bounded too.

**Proposition 2.2** Given \( C \) in \( \mathbb{R}^2 \) such that \( C_1 C_2 \neq 0 \), the following properties hold:

(i) \( E \) is coercive;

(ii) critical points of \( E \) over \( M_C \) are solutions to the elliptic system (1.3).
(iii) If \((u, \omega)\) is a minimiser, then for \(j = 1, 2\), \(u_j\) is either positive or negative.

**Proof.** The proof of (i) follows from the arguments used in Step I of [3, Proof of Lemma 2.7].

(ii) If \((u, \omega)\) is a critical point, there are two Lagrange multipliers \(\lambda_1\) and \(\lambda_2\) such that

\[
DE = \lambda_1 DC_1 + \lambda_2 DC_2.
\]

Taking the projection on \(H^1(\mathbb{R}^N, \mathbb{R}^2) \times \{0\}\), and on \([0] \times \mathbb{R}^2\), we obtain

\[
-\Delta u_j + m_j^2 u_j + \partial_z F(u) = 2\lambda_j \omega_j u_j,
\]

\[
\omega_j \|u_j\|^2_{L^2} = \lambda_j \|u_j\|^2_{L^2}
\]

for \(j = 1, 2\). Because \(u_j \neq 0\) we obtain \(\lambda_j = \omega_j\) and thus (1.3). By local regularity results, [13, §8], \(u\) is a classic solution.

(iii) We define \(w_j := |u_j| \geq 0\). From (A3) it follows that \((w, \omega) \in M_C\) and \(E(u, \omega) = E(w, \omega)\). By (ii),

\[
-\Delta w_j = (\omega_j^2 - m_j^2)w_j + \gamma \omega_j^2 w^\gamma_j \partial\sigma_j G(w)
\]

where \(\sigma(1) = 2\) and \(\sigma(2) = 1\). Hence,

\[
-\Delta w_j + \lambda_j w_j + \partial\sigma_j G(w) \geq 0,
\]

where \(\lambda_j := m_j^2 - \omega_j^2\). Let us define

\[
A_j(x) = \begin{cases} 
\lambda_j + \partial\sigma_j G(w)^{-1} & \text{if } w_j(x) \neq 0 \\
\lambda_j & \text{otherwise.}
\end{cases}
\]

By (A2), and the continuity of \(w_j\) and \(\partial\sigma_j G\), we have \(A_j^+ \in C_\nu(\mathbb{R}^N)\). Therefore,

\[
-\Delta w_j + A_j^+(x)w_j \geq 0.
\]

Now, we can apply the strong maximum principle. Hence, \(w_j > 0\).

### 3 The sub-additivity property of \(I\)

Given a non-negative function \(f\), we denote with \(f^\ast\) the Steiner symmetrization with respect to the direction \(e\) in \(\mathbb{R}^N\) (with \(|e| = 1\), [18, §3.7, p. 87]. We denote with \(f^\ast\) the symmetric rearrangement, [18, §3.3, p. 80]. The next lemma addresses the one-dimensional case of [10, Proposition 1.4]. The argument goes back to B. Kawohl [17, Lemma 2.6, p. 33].

**Lemma 3.1** Let \(u, v \in H^1(\mathbb{R})\) non-negative functions with compact support, symmetric and radially decreasing with respect to the origin, and such that \(u(0) \leq v(0)\). Let \(T\) be such that

\[
\text{supp}(u) \cap \text{supp}(v(-T)) = \emptyset.
\]

Then

\[
\|w^\ast\|^2_{L^2} \leq \|w'\|^2_{L^2} - \frac{2}{3}\|u'\|^2_{L^2}
\]

where \(w(t) := u(t) + v(t - T)\).
Proof. We denote with \([-c, c]\) and \([-d, d]\) the support of \(u\) and \(v\), respectively. Firstly, we prove the estimate under the additional assumptions that \(u\) and \(v\) are continuously differentiable and

\[
\begin{align*}
tu'(t) &< 0 \text{ on } \{t \in (-c, c), t \neq 0\} \\
tv'(t) &< 0 \text{ on } \{t \in (-d, d), t \neq 0\}.
\end{align*}
\]

We set \(a := \sup(u)\) and \(b := \sup(v)\). The functions

\[
u: [0, c] \to [0, a], \quad v: [0, d] \to [0, b]
\]

are invertible because they are strictly decreasing. Their inverses, \(y_u\) and \(y_v\), are continuously differentiable on \((0, a)\) and \((0, b)\), respectively. Thus,

\[
u(y_u(s)) = s \text{ on } [0, a], \quad v(y_v(s)) = s \text{ on } [0, b].
\]

Because \(w^*\) is symmetric and decreasing, the level set \(\{w^* \geq s\}\) is an interval. We denote its length by \(2z(s)\). We have

\[
2z(s) = \|w^* \geq s\| = \begin{cases} \int_0^a y_u(s) + 2y_v(s) & \text{if } s \in [0, a] \\ 2y_v(s) & \text{if } s \in [a, b]. \end{cases}
\]

Thus, \(z\) is strictly decreasing and continuously differentiable for every \(s \notin [0, a, b]\). Moreover,

\[
w^*(z(s)) = s \text{ on } [0, b],
\]

Taking the derivative with respect to \(s\) in (5.5) and in (5.3), we have

\[
w^{**}(z(s))z'(s) = 1, \quad u'(y_u(s))y_u'(s) = 1, \quad v'(y_v(s))y_v'(s) = 1
\]

on the complement of a finite set. Hence,

\[
\begin{align*}
\int_R |w^{**}|^2 dt &= 2 \int_0^a |w^*|^2 dt = -2 \int_0^a |w^*(z(s))|^2 z'(s) ds = -2 \int_0^a (z'(s))^2 ds \\
&= -2 \int_0^a (y_v(s) + y_v'(s))^{-1} ds - 2 \int_a^b (y_v'(s))^{-1} ds.
\end{align*}
\]

The second equality follows from a change of variable and the first of (5.6). The fourth equality follows from (5.4). From the inequality

\[
2(x + y)^{-1} \leq x^{-1} + y^{-1} - \max\{x^{-1}, y^{-1}\}, \quad x, y > 0
\]

the first integration of the second line of (5.7) can be estimated from above with

\[
-\frac{1}{2} \int_0^a (y_v'(s))^{-1} ds + \frac{1}{2} \int_0^a \max\{y_u'(s), y_v'(s)\} ds.
\]

Using the estimate \(2 \max\{t, s\} \geq t + s\), (5.8) and (5.7), the left member of the first equality in (5.7) is bounded by

\[
\begin{align*}
&-\frac{1}{2} \int_0^a (y_u'(s))^{-1} ds - \frac{1}{2} \int_0^a (y_v'(s))^{-1} ds - 2 \int_a^b (y_v'(s))^{-1} ds \\
&\leq -\frac{1}{2} \int_0^a (y_u'(s))^{-1} ds - 2 \int_a^b (y_v'(s))^{-1} ds \\
&= \frac{1}{4} \left( -2 \int_0^a (y_u'(s))^{-1} ds + \left( -2 \int_0^b (y_v'(s))^{-1} ds \right) \right).
\end{align*}
\]
From a change of variable and (5.6), it follows that
\[ \|u''\|_{L^2}^2 = -2 \int_0^\infty (y''(s))^{-1} ds, \quad \|v''\|_{L^2}^2 = -2 \int_0^\infty (y''(s))^{-1} ds. \]
Thus, from (5.7), we obtain
\[ \\|w''\|_{L^2}^2 \leq \frac{1}{4} \|u''\|_{L^2}^2 + \|v''\|_{L^2}^2 = \|w''\|_{L^2}^2 - \frac{3}{4} \|u''\|_{L^2}^2. \]  
(5.9)

In the general case, we can approximate \( u \) and \( v \) with functions satisfying (5.1) and (5.2): firstly, we consider
\[ \sigma_u : [0, c] \to \mathbb{R}^+, \quad \sigma_u'(t) < 0 \text{ on } (0, c), \quad \sigma_u(0) = 0 \]  
(5.10)
smooth, and extend it to \( \mathbb{R} \) as \( \sigma_u(-t) = \sigma_u(t) \). We define
\[ U := u + \|u - v\|_{L^\infty(0,\delta)} \sigma_u, \quad u_\delta := \rho_\delta * U \]  
(5.11)
where \( \rho_\delta \) is a symmetric mollifier. Thus, \( u_\delta \) is an even function. Because \( U \) is strictly decreasing, given \( t \geq 0 \), we have
\[ u_\delta'(t) = \int_0^\delta \rho_\delta'(y)(U(t - y) - U(t + y))dy < 0, \]
unless \( t = 0 \). Similarly, we define \( \sigma_v \) as in (5.10) with the additional hypothesis
\[ \sigma_u(0) < \sigma_v(0) - 1. \]  
\( V \) and \( v_\delta \) are defined as in (5.11), by replacing \( \sigma_u \) with \( \sigma_v \). Thus, if \( \delta > 0 \) is sufficiently small,
\[ \sup(u_\delta) \leq \sup(v_\delta) \]
and the supports of \( u_\delta \) and \( v_\delta(-\cdot - T) \) are disjoint. Therefore, we can apply estimate (5.9) to
\[ w_\delta = u_\delta + v_\delta(-\cdot - T) \]
and obtain
\[ \|w_\delta''\|_{L^2}^2 \leq \|w''_\delta\|_{L^2}^2 - \frac{3}{4} \|u''_\delta\|_{L^2}^2. \]
By the continuity of the symmetric rearrangement in \( H^1(\mathbb{R}) \), [12], we can take the limit as \( \delta \to 0 \) in the above inequality.

**Proposition 3.1** Let \( \rho, \tau \) be such that \( \rho_j \geq \tau_j > 0 \) and \( \tau \neq \rho \). Then,
\[ I(\rho) < I(\tau) + I(\rho - \tau). \]

**Proof.** Define \( \sigma := \rho - \tau \), and let
\[ (u_\delta)_{\delta \geq 1} \subset N_\tau, \quad (v_\delta)_{\delta \geq 1} \subset N_\sigma \]  
(5.12)
be minimising sequences of \( J \) over \( N_\tau \) and \( N_\sigma \), respectively. By (iii) of Proposition 2.1, we can suppose that each of the sequences have compact support, that \( u_\delta' \) and \( v_\delta' \) are non-negative, from (A3), and symmetrically decreasing, by (A1), (A2), [26, Lemma 1] and [18, Theorem 3.4, p. 82].

We set \( e_N := (0, \ldots, 0, 1) \). Let \((T_n)_{n \geq 1}\) be a real sequence such that the two functions
\[ u_\delta', v_\delta'(- \cdot + T_n e_N) \]
have disjoint support for every $i, j$ in $[1, 2]$. Then,

$$w_n := u_n + v_n(\cdot + T_n e_N) \in N_0$$  \hspace{1cm} (5.13)

$$J(w_n) = J(u_n) + J(v_n).$$  \hspace{1cm} (5.14)

We denote the Steiner symmetrization of $w_n$ with respect to $e_N$ with $w_n^{**}$. By [17, (C), p. 22], $w_n^{**} \in N_0$. From [18, (v), p. 81], and [18, Eq. (1), p. 82],

$$-\int_{\mathbb{R}^N} |w_n^{**}| \, dx \leq -\int_{\mathbb{R}^N} |w_n^{**}|^\gamma \, dx.$$  

Along with $(A_4)$, the above inequality yields

$$\int_{\mathbb{R}^N} F(w_n^{**}) \leq \int_{\mathbb{R}^N} F(w_n).$$

By [26, Lemma 1],

$$||Dw_n^{**}||_{L^2} \leq ||Dw_n||_{L^2}. \hspace{1cm} (5.15)$$

Thus $J(w_n^{**}) \leq J(w_n)$. Given $x' \in \mathbb{R}^{N-1}$,

$$\partial_{x_n} w_n^{**}(x', t) = w_n^{**}(x', \cdot)'(t).$$

Then, we can write

$$\int_{\mathbb{R}^N} |\partial_{x_n} w_n^{**}|^2 \, dx = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} |w_n^{**}(x', \cdot)'(t)|^2 \, dt \, dx'$$

$$= \int_{U'_2} \int_{\mathbb{R}} |w_n^{**}(x', \cdot)'(t)|^2 \, dt \, dx'$$

$$+ \int_{V'_2} \int_{\mathbb{R}} |w_n^{**}(x', \cdot)'(t)|^2 \, dt \, dx' = A'_1 + A'_2$$

where

$$U'_2 = \{x' \in \mathbb{R}^{N-1} \mid \sup_{\mathbb{R}} u_n'(x', \cdot) \leq \sup_{\mathbb{R}} v_n'(x', \cdot)\}$$

$$V'_2 = \{x' \in \mathbb{R}^{N-1} \mid \sup_{\mathbb{R}} v_n'(x', \cdot) < \sup_{\mathbb{R}} u_n'(x', \cdot)\}.$$  

For every $x' \in \mathbb{R}^{N-1}$, $u_n'(x', \cdot)$ and $v_n'(x', \cdot)$ satisfy the hypotheses of Lemma 3.1 with $T = T_n$. Thus,

$$A'_1 \leq \int_{U'_2} (||w_n^{**}(x', \cdot)'||^2_{L^2(\mathbb{R})} - \frac{3}{4} ||u_n'(x', \cdot)'||^2_{L^2(\mathbb{R})}) \, dx'$$

$$A'_2 \leq \int_{V'_2} (||w_n^{**}(x', \cdot)'||^2_{L^2(\mathbb{R})} - \frac{3}{4} ||v_n'(x', \cdot)'||^2_{L^2(\mathbb{R})}) \, dx'.$$

Taking the sum, we obtain

$$A'_1 + A'_2 \leq ||\partial_{x_n} w_n||^2_{L^2} - \frac{3}{4} \left(||\partial_{x_n} u_n||^2_{L^2(U'_2 \cap \mathbb{R}^{N-1})} + ||\partial_{x_n} v_n||^2_{L^2(V'_2 \cap \mathbb{R}^{N-1})}\right).$$

Because $u_n'$ and $|\partial_{x_n} u_n'|$ are radially symmetric, we have

$$||Du_n'||^2_{L^2(U'_2 \cap \mathbb{R}^{N-1})} = N ||\partial_{x_n} u_n||^2_{L^2(U'_2 \cap \mathbb{R}^{N-1})},$$
From (5.13), it follows that

$$N\|\partial_{x_n} w_n^j\|_{L^2}^2 = \|D v_n^j\|_{L^2}^2.$$  

Thus,

$$N(A_j^1 + A_j^2) \leq \|D v_n^j\|_{L^2}^2 - \frac{3}{4} \left( \|D u_n^j\|_{L^2(U_n^j \times B^R_n)}^2 + \|D v_n^j\|_{L^2(V_n^j \times B^R_n)}^2 \right),$$  

(5.17)

We define

$$d_n^j = \|D u_n^j\|_{L^2(U_n^j \times B^R_n)}^2 + \|D v_n^j\|_{L^2(V_n^j \times B^R_n)}^2.$$  

We prove that $(d_n^j)_{n \geq 1}$ is bounded from below. On the contrary, up to extract a subsequence, we can suppose that $d_n^j \to 0$ for some $1 \leq j \leq 2$. Because $u_n$ and $v_n$ are minimising sequences, by (ii) of Proposition 2.1, they are also bounded in $H^1$. By construction, $u_n$ and $v_n$ are radially decreasing. Then, by [5, Theorem A.1'], up to extract a subsequence, we can suppose that

$$u_n^j \to u_j, v_n^j \to v_j \text{ in } L^{2}\left(\mathbb{R}^N\right), \text{ a.e.}$$  

By (i) of Proposition 2.1 and the first inequality in (3.2)

$$\|u_n^j\|_{L^2} \geq \|v_n^j\|_{L^2} \geq C = c(\rho, \tau) > 0,$$  

(5.18)

whence $u_j, v_j \not\equiv 0$. We fix $R > 0$ and consider the domains

$$E_n^j := (U_n^j \times \mathbb{R}) \cap B_R, \quad F_n^j := (V_n^j \times \mathbb{R}) \cap B_R.$$  

(5.19)

Because the two domains are bounded,

$$d_n^j \geq \frac{1}{m(E_n^j)} \cdot \|D u_n^j\|_{L^2(E_n^j)}^2 + \frac{1}{m(F_n^j)} \cdot \|D v_n^j\|_{L^2(F_n^j)}^2 \geq \frac{1}{\omega_N R^N} \left( \|D u_n^j\|_{L^2(E_n^j)}^2 + \|D v_n^j\|_{L^2(F_n^j)}^2 \right).$$  

(5.20)

Up to extract a subsequence there are two sets $U_j, V_j \subset \mathbb{R}^{N-1}$ such that the convergence

$$\chi_{U_n^j} \to \chi_{U_j}, \quad \chi_{V_n^j} \to \chi_{V_j}$$  

is strong in $L^2(B_R^{N-1})$, where $B_R^{N-1} := B_R \cap (\mathbb{R}^{N-1} \times 0)$. Moreover, $U_j$ and $V_j$ are radially symmetric and the convergence

$$\chi_{E_n^j} \to \chi_{E_j}, \quad \chi_{F_n^j} \to \chi_{F_j}$$  

is strong in $L^2(B_R)$, where

$$E_j = (U_j \times \mathbb{R}) \cap B_R, \quad F_j = (V_j \times \mathbb{R}) \cap B_R.$$  

Taking the limit in (5.20), we obtain

$$Du_j \equiv 0, \quad E_j \text{ a.e.,} \quad Dv_j \equiv 0, \text{ on } F_j \text{ a.e.}$$  

whence

$$Du_j \equiv 0 \text{ on } U_j, \quad Dv_j \equiv 0 \text{ on } V_j$$  

(5.21)

and

$$u_j \leq v_j \text{ on } U_j, \quad v_j \leq u_j \text{ on } V_j,$$  

(5.22)
By the Ekeland Principle, we can suppose that the sequences in (5.12) are Palais-Smale. Therefore, $u_j$ and $v_j$ are weak solutions to an elliptic system and, by local regularity results, continuously differentiable. Thus, we can suppose that $U_j$ is open and $V_j$ is closed. Because such sets are radially symmetric, we can write

$$U_j = \{ x' \in B_R^{N-1} | |x'| \in \Omega \}, \quad V_j = \{ x' \in B_R^{N-1} | |x'| \in G \}$$

where $\Omega$ and $G$ are open and closed subsets of $\langle e_1 \rangle$. We set

$$\Omega_1 := \Omega \cap \{ t \epsilon_1 | t > 0 \}, \quad G_1 := G \cap \{ t \epsilon_1 | t > 0 \}.$$ 

Then

$$\Omega_1 = \bigcup_{i \in \mathbb{Z}} (a_i, b_i), \quad a_i \leq b_i, \quad G_1 = \bigcup_{i \in \mathbb{Z}} [b_i, a_{i+1}].$$

For every $i \in \mathbb{Z}$, $v_j$ is constant on $[b_i, a_{i+1}]$ by (5.21). Thus,

$$v_j(b_i) = v_j(a_{i+1}). \quad (5.23)$$

In the case $b_i = a_{i+1}$ the above equality is obviously true. By the continuity of $u_j$ and $v_j$, and (5.22) and (5.21), it follows

$$u_j(b_i) = v_j(b_i), \quad u_j(a_{i+1}) = v_j(a_{i+1})$$

$$u_j \equiv c_i \text{ on } (a_i, b_i) \quad (5.24)$$

for some constant $c_i \in \mathbb{R}$. From (5.23) and (5.24) we have

$$c_i = u_j(b_i) = v_j(b_i) = v_j(a_{i+1}) = u_j(a_{i+1}) = c_{i+1}.$$ 

Given $x \in [b_i, a_{i+1}]$

$$c_i \geq u_j(x) \geq c_{i+1} = c_i,$$

because $u_j$ is monotonically non-increasing. Then, $u_j$ is constant on $\{ t \epsilon_1 | t > 0 \}$. Because $u_j$ is radially symmetric, $u_j$ is constant on $B_R$. By applying the same argument for every $R > 0$, we obtain that $u_j$ is constant on $\mathbb{R}^N$. Because $u_j$ is $L^2$, we have $u_j \equiv 0$ obtaining a contradiction with (5.18). The contradiction follows from the assumption that $d_n^j \rightarrow 0$. So, we proved that each of the sequences $(d_n^j)_{n \geq 1}$ is bounded from away from zero. Let $d$ be such that

$$d_n^j \geq d \text{ for all } n.$$ 

Then, from (5.16), (5.17) we obtain

$$N \int_{\mathbb{R}^n} |\partial_{x_n} w_n^{j \epsilon \nu} |^2 dx \leq \| Dw_n^{j \epsilon \nu} \|^2_{L^2} - \frac{3d_n^j}{4} \leq \| Dw_n^{j \epsilon \nu} \|^2_{L^2} - \frac{3d}{4}. \quad (5.26)$$

Finally, we consider the decreasing rearrangement of $w_n^{j \epsilon \nu}$. By applying (5.15) in dimension $N = 1$, we have

$$\| \partial_{x_n} w_n^{j \epsilon \nu} \|^2_{L^2} = \int_{\mathbb{R}^{N-1}} |w_n^{j \epsilon \nu} (x', \cdot') |^2_{L^2(\mathbb{R})} dx' \leq \int_{\mathbb{R}^{N-1}} |w_n^{j \epsilon \nu} (x', \cdot') |^2_{L^2(\mathbb{R})} dx' = \| \partial_{x_n} w_n^{j \epsilon \nu} \|^2_{L^2}.$$ 

From (5.26), we note

$$N \int_{\mathbb{R}^n} |\partial_{x_n} w_n^{j \epsilon \nu} |^2 \leq \| Dw_n^{j \epsilon \nu} \|^2_{L^2} + \| Dw_n^{j \epsilon \nu} \|^2_{L^2} - \frac{3d}{4}. \quad (5.27)$$
Because $w_n^{j\star e^N}$ is radially symmetric, from (5.26) it follows that
\[
\int_{\mathbb{R}^N} |Dw_n^{j\star e^N}|^2 \leq ||Du_n|^2_{L^2} + ||Dv_n|^2_{L^2} - \frac{3d}{4}
\]
and
\[
J(w_n^{j\star e^N}) \leq J(w_n^{e^N}), \quad w_n^{j\star e^N} \in N_p.
\]
Hence,
\[
I(\rho) \leq J(w_n^{j\star e^N}) \leq J(u) + J(v) - \frac{3d}{4}
\]
Taking the limit as $n \to \infty$, we obtain
\[
I(\rho) \leq I(\tau) + I(\sigma) - \frac{3d}{4}.
\]
We set $D := 3d/4 > 0$.

4 Minimising sequences of $(J, N_p)$ and $(E, M_C)$

Lemma 4.1 Let $(u_n)_{n \geq 1}$ be a bounded sequence in $H^1$ such that
\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^\gamma > 0
\]
where $1 < \gamma < 2^*/2$. Then, there exist $u \in H^1$ and a sequence $(y_n)_{n \geq 1} \subset \mathbb{R}^N$ such that
\[
u_n := u_n(u - y_n) \to u_j, \quad u_1u_2 \neq 0.
\]
Proof. Let $w_n = u_n^1u_n^2$. From the Schwarz inequality, we have
\[w_n \in L^1(\mathbb{R}^N),\]
by applying the H"older inequality with the pair of exponents
\[\left(\frac{2(N-1)}{N}, \frac{2(N-1)}{N-2}\right),\]
we obtain
\[Dw_n \in L^{N/(N-1)}(\mathbb{R}^N).
\]
We use [20, Lemma I.1] with $q = 1$ and $p = N/(N-1)$. Hence, given $R > 0$, either there exists a sequence $(y_n)_{n \geq 1}$ such that
\[
\liminf_{n \to \infty} \int_{B(\cdot - y_n, R)} |w_n| > 0 \quad (6.1)
\]
or
\[w_n \to 0 \text{ in } L^\alpha(\mathbb{R}^N), \quad \alpha \in (1, N/(N-2)).\]
The latter is ruled out by the hypothesis on $\gamma$. Hence, (6.1) holds. We set
\[v_n := u_n(u - y_n)\]
and obtain
\[
\liminf_{n \to \infty} \int_{B_R} |v_n|^2 > 0. \quad (6.2)
\]
Let $G \geq 0$. Theorem 4.1 suggests that the two above inequalities are equalities:

$$\int_{B_R} u_1 u_2 > 0$$

which implies $u_1 u_2 \neq 0$.

**Theorem 4.1** Let $(u_n)_{n \geq 1}$ be a minimising sequence for $J$ over $N_\rho$. Then, there exists $u \in N_\rho$ and a sequence $(y_n)_{n \geq 1}$ such that

$$u_n = u(\cdot + y_n) + o(1) \text{ in } H^1$$

$$J(u) = \inf_{N_\rho} J.$$

**Proof.** By (i) and (ii) of Proposition 2.1, $I(\rho) < 0$ and the sequence $(u_n)_{n \geq 1}$ is bounded. Because $G \geq 0$, $(u_n)_{n \geq 1}$ fulfils the hypothesis of Lemma 4.1 if $\gamma < N/(N - 2)$ holds. This, in turn, follows from $(A_1)$ and

$$1 + \frac{2}{N} < \frac{N}{N - 2}.$$

Then, we consider the sequence $(y_n)_{n \geq 1}$ and $u \in H^1$ given by Lemma 4.1. We define

$$v_n := u_n(\cdot - y_n) - u, \quad \tau := (\|u_1\|_2^2, \|u_2\|_2^2).$$

Note that $\tau_j \leq \rho_j$ by the weak lower semi-continuity property of the $L^2$-norm and that $\tau_j > 0$, from Lemma 4.1. Suppose that $\tau \neq \rho$. By (iv) of Proposition 2.1, up to extract a subsequence, we can suppose that

$$J(v_n) = J(u_n(\cdot - y_n)) - J(u) + o(1).$$

After a change of variable, the first term of the right member equals $J(u_n)$, which converges to $I(\rho)$. Hence, by Proposition 3.1

$$I(\rho - \tau) \leq I(\rho) - I(\tau) < I(\rho - \tau).$$

Thus, we obtain a contradiction with the assumption that $\tau \neq \rho$. Then $\tau = \rho$ and $u \in N_\rho$. Thus,

$$u_n(\cdot - y_n) - u_j \to 0 \text{ in } L^2(\mathbb{R}^N).$$

Up to extract a subsequence, we can suppose that the above convergence is weak in $H^1$. We set $w_n := u_n(\cdot - y_n)$. By (3.1), the above convergence holds in $L^2(\mathbb{R}^N)$ and $L^4(\mathbb{R}^N)$. Therefore, as in the proof of (iii) of Proposition 2.1, we conclude that

$$\int_{\mathbb{R}^N} F(w_n) \to \int_{\mathbb{R}^N} F(u).$$

We have

$$J(w_n) = \int_{\mathbb{R}^N} F(w_n) + \frac{1}{2} \int_{\mathbb{R}^N} |Dw_n|^2 \geq \int_{\mathbb{R}^N} F(u) + \frac{1}{2} \int_{\mathbb{R}^N} |Du|^2 = J(u).$$

Because $(w_n)_{n \geq 1}$ is a minimising sequence, taking the limit, we obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |Dw_n|^2 = \int_{\mathbb{R}^N} |Du|^2, \quad J(u) = I(\rho).$$

Then, the above inequalities are equalities:

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |Dw_n|^2 = \int_{\mathbb{R}^N} |Du|^2, \quad J(u) = I(\rho).$$
Thus, \( Dw_n \to Du \) strongly in \( L^2 \) and \( u \) is a minimiser.

**Proof of Theorem 1.1** From (i) of Proposition 2.2, given a minimising sequence \((u_n, \omega_n)\), there exists \( \rho \) such that

\[
\|u_n\|_{L^2} \to \sqrt{\rho}, \quad \omega_n \to \omega.
\]

As in Step II of the proof of [3, Lemma 2.7], it can be shown that

\[
v_j = \frac{\sqrt{\rho} u_j}{\|u_j\|_{L^2}}
\]

is a minimising sequence for \( J \) over \( N_{\rho} \) (notice that, unlike stated in [3, p. 13], their proof requires only a combined power-type estimate on \( DF \), as in (A_2), rather than the condition (H_3) of [3]). Then, by Theorem 4.1, there exists a sequence \((y_n)_{n \geq 1} \subset \mathbb{R}^N\) such that

\[
v_n(\cdot + y_n) \to u \text{ in } H^1
\]

for some \( u \in H^1 \). Then, \((u, \omega) \in M_C\) is a minimiser of \( E \) over \( M_C \).

## 5 Stability results

**Lemma 5.1** Let \( \phi \) be a \( H^1(\mathbb{R}^N, \mathbb{R}^k) \) function. Then \( |\phi| \) is \( H^1(\mathbb{R}^N) \) and

\[
\|D|\phi||_{L^2} \geq \|D|\phi||_{L^2}. \tag{7.1}
\]

Suppose that for every bounded subset \( S \subset \mathbb{R}^N \) ess inf \( S |\phi| > 0 \). If equality holds between the two above norms, then there exists \( \lambda \) in \( \mathbb{R}^k \) such that \( |\lambda| = 1 \) and

\[
\phi(x) = \lambda |\phi(x)|. \tag{7.2}
\]

**Proof.** The proof of the fact that \( |\phi| \) is \( H^1(\mathbb{R}^N, \mathbb{R}^k) \) follows the same steps of the case \( k = 2 \) in [18, Theorem 6.17, p. 152]. Then

\[
\partial_x |\phi| = \begin{cases} (\phi, \partial_x \phi) / |\phi| & \text{if } \phi \neq 0 \\ 0 & \text{if } \phi = 0 \end{cases}
\]

for every \( 1 \leq i \leq N \). By the Schwarz inequality,

\[
|D|\phi|^2 = \sum_{i=1}^N |\partial_x |\phi||^2 = \frac{1}{|\phi|^2} \sum_{i=1}^N (\phi, \partial_x \phi)^2 \leq \sum_{i=1}^N |\partial_x \phi|^2 = |D\phi|^2 \tag{7.3}
\]

if \( \phi \neq 0 \). On the region \( |\phi| = 0 \), the same inequality follows easily. Then \( D|\phi| \) is \( L^2 \). By integrating (7.3), we prove the first part of the statement. Now, we suppose that in (7.1) the equality holds and \( |\phi| \) is essentially bounded from below on every bounded subset of \( \mathbb{R}^N \). From (7.3) we obtain

\[
|\phi| |\partial_x \phi| = (\phi, \partial_x \phi).
\]

Because \( \phi(x) \neq 0 \) a.e., there exists \( \mu_i : \mathbb{R}^N \to \mathbb{R} \) such that

\[
\partial_x \phi = \mu_i \phi \text{ a.e.} \tag{7.4}
\]

We claim that each of the functions

\[
\Lambda_j : \mathbb{R}^N \to \mathbb{R}, \quad x \mapsto \frac{\phi_j(x)}{|\phi(x)|}
\]
is constant. From the same approximation argument as [18, Theorem 6.16, p. 178], it follows that $\Lambda_j$ is $H^1_{loc}(\mathbb{R}^N)$ and

$$|\phi|^2 \partial_x \Lambda_j = \partial_x \phi |\phi|^2 - \phi_j(\phi, \partial_x \phi) = \sum_{b=1}^k \partial_x \phi_b \phi_b^2 - \phi_j \phi_0 \partial_x \phi_b = \mu |\phi|^2 - \phi_j \phi_b^2 = 0.$$ 

The last equality follows from (7.4). So, there exists $\lambda_j$ in $\mathbb{R}$ with $\Lambda_j \equiv \lambda_j$ a.e. which satisfies (7.2).

A similar result has been proved in [18, Theorem 7.8] in the case $k = 2$, under the assumption that one of the components of $\phi$ is positive almost everywhere.

Let $C$ be such that $C_j \neq 0$ for $j = 1, 2$. For every $(\phi, \psi)$ in $X$ such that $\phi_j \neq 0$, we define the map

$$X \ni (\phi, \psi) \mapsto P(\phi, \psi) := \left[|\phi_1|, |\phi_2|, \frac{C_1}{||\phi_1||^2}, \frac{C_2}{||\phi_2||^2} \right] \in M_C.$$ 

(7.5)

**Proposition 5.1** For every $\Psi := (\phi, \psi)$ such that $\phi_j \neq 0$, for $j = 1, 2$, there holds

$$E(\Psi) \geq E(P(\Psi)), \quad C_j(\Psi) = C_j(P(\Psi)).$$

In the proposition $E$ and $C_j$ are the energy and charges defined in (1.6) and (1.7).

**Proof:** From the Schwarz inequality, we obtain

$$\frac{|C_j(\phi, \psi)|}{||\phi||^2} \leq ||\phi||^2.$$ 

(7.6)

By Lemma 5.1 and (7.6),

$$E(\phi, \psi) = \frac{1}{2} \int_{\mathbb{R}^N} |D\phi|^2 + |\phi|^2 + 2V(\phi) \geq \frac{1}{2} \int_{\mathbb{R}^N} |D\phi|^2 + 2V(|\phi_1|, |\phi_2|) + \frac{1}{2} \sum_{j=1}^2 \frac{C_j(\phi, \psi)^2}{||\phi||^2}.$$ 

$$= E(P(\Psi)),$$

and

$$C_j(P(\Psi)) = \frac{C_j(\Psi)}{||\phi||^2} \int_{\mathbb{R}^N} |\phi|^2 = C_j(\Psi).$$

**Proof of Theorem 1.2.** Given $\Phi$ in $\Gamma_C$ there exists $(u, \omega)$ in $K_C$ and $(\lambda, y)$ in $\mathbb{R}^2 \times \mathbb{R}^N$ such that

$$\Phi = (\lambda \cdot u( + y), -i \omega \cdot \lambda \cdot u( + y)).$$

We used the notation introduced in (1.2). Then

$$E(\Phi) = E(u, \omega) = m_C, \quad C_j(\Phi) = \omega_j ||u||^2 = C_j.$$

Because $E$ and $C_j$ are continuous, if $d(\Phi_n, \Gamma_C) \to 0$, then

$$E(\Phi_n) \to m_C, \quad C_j(\Phi_n) \to C_j.$$ 

(7.7)

We prove the converse and suppose that (7.7) holds. We set

$$\Phi_n := (\phi_n, \psi_n).$$
Because $C_j \neq 0$ for $j = 1, 2$, $\phi^j_n \neq 0$ for $n$ large enough. Then, it makes sense to define

$$\phi^j_n := \Phi^j \Phi_n.$$  

(7.8)

From Proposition 5.1 and (7.7), $(u_n, \omega_n)$ is a minimising sequence of $E$ over $M_C$. By Theorem 1.1, there are

$$(u, \omega) \in K_C, \ (y_n)_{n \geq 1} \subset \mathbb{R}^N$$

such that

$$u_n = u(\cdot + y_n) + o(1), \ \omega_n = \omega + o(1).$$  

(7.9)

We set

$$\psi_n := \phi_n(\cdot - y_n), \ \psi^j_n := \phi^j_n(\cdot - y_n).$$

By a change of variable, we have

$$E(\psi_n, \psi^j_n) = E(\phi_n, \phi^j_n), \ \ C_{j}(\psi_n, \psi^j_n) = C_{j}(\phi_n, \phi^j_n).$$  

(7.10)

Up to extract a subsequence, we can suppose that there exists $(\psi, \psi_j)$ in $X$ such that

$$\psi_n \rightharpoonup \psi \text{ in } H^1(\mathbb{R}^N, \mathbb{C}), \ \psi^j_n \rightarrow \psi_j \text{ in } L^2(\mathbb{R}^N, \mathbb{C}^2).$$  

(7.11)

By the weak lower semi-continuity of the norm, the strong convergence of $|\psi_n|$, (7.6) and Lemma 5.1, we have

$$E(\psi_n, \psi^j_n) = \frac{1}{2} \int_{\mathbb{R}^N} |D\psi_n|^2 + |\psi^j_n|^2 + 2V(\psi_n)$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^N} |D\psi|^2 + |\psi_j|^2 + 2V(\psi)$$

$$\geq \frac{1}{2} \int_{\mathbb{R}^N} |D\psi|^2 + 2V(|\psi_1|, |\psi_2|) + \frac{1}{2} \sum_{j=1}^2 \frac{C_j(\psi, \psi_j)^2}{||\psi_j||^2} \geq mc.$$  

Taking the limit as $n \to \infty$, by (7.10), the first of (7.7) and the first of the above inequalities, we obtain

$$\lim_{n \to \infty} ||\psi_n||^2 = ||\psi||^2, \ \lim_{n \to \infty} ||D\psi_n||^2 = ||D\psi||^2.$$  

(7.12)

From the second inequality, we obtain

$$\int_{\mathbb{R}^N} |D\psi_j|^2 = \int_{\mathbb{R}^N} |D\psi|^2, \ \frac{C_{j}(\psi, \psi_j)}{||\psi_j||^2} = ||\psi_j||^2.$$  

(7.13)

The weak limit in (7.11) and the strong convergence of $|\psi^j_n|$ to $u_j$ implies that

$$|\psi_j| = u_j \text{ a.e.}$$  

(7.14)

and

$$\psi^j_n \rightharpoonup \psi_j \text{ in } L^2(\mathbb{R}^N, \mathbb{C}).$$  

(7.15)

Because $(u, \omega)$ is a minimiser of $E$ over $M_C$, $u_j$ are regular, by (ii), and positive, by (iii) of Proposition 2.2 and (7.14). Thus, $\psi_j$ fulfils the hypotheses of Lemma 5.1. From (7.13) there are $\lambda_j$ in $\mathbb{C}$ such that $|\lambda_j| = 1$ and

$$\psi_j = \lambda_j |\psi_j| = \lambda_j u_j.$$  

The second limit in (7.12) and the first in (7.11) yield

$$D\psi^j_n \rightarrow D\psi_j.$$
By (7.15),
\[ \psi'^{j}_{n} \rightarrow \lambda_{j} u_{j} \text{ in } H^{1}(\mathbb{R}^{N}, \mathbb{C}). \tag{7.16} \]
The second equality in (7.13) can be written as
\[ \text{Re} \int_{\mathbb{R}^{N}} -i \psi_{j} \cdot \psi_{j} = \|\psi_{j}'\|_{L^{2}}^{2} \|\psi_{j}\|_{L^{2}}. \]
Thus, we have an equality between the scalar product and the product of norms. Then
\[ \psi_{j}' = -i C_{j} \||\psi_{j}'\|_{L^{2}}^{2} \psi_{j}. \tag{7.17} \]
From (7.5) and (7.8), we have
\[ \omega_{j}^{l} = \frac{C_{j}}{\|\psi_{j}''\|_{L^{2}}^{2}}. \]
Taking the limit, we obtain
\[ \omega_{j} = \frac{C_{j}}{\|\psi_{j}''\|_{L^{2}}^{2}}. \]
Then (7.17) can be written as
\[ \psi_{j}' = -i \omega_{j} \psi_{j}. \]
By the second limit in (7.11) and the first limit of (7.12)
\[ \psi_{n}^{l} \rightarrow \psi_{j}' = -i \omega_{j} \lambda_{j} u_{j} \text{ in } L^{2}(\mathbb{R}^{N}, \mathbb{C}). \tag{7.18} \]
Thus, (7.16) and (7.18) yield
\[ d((\psi_{n}', \psi_{n}''), \Gamma_{C}) \rightarrow 0 \]
so that \( d((\phi_{n}', \psi_{n}''), \Gamma_{C}) \rightarrow 0. \)

**Proof of Theorem 1.3.** The proof of the stability of \( \Gamma_{C} \) follows from the fact that \( \mathbf{V} \), defined in (1.9), is a Lyapunov function (see [3, Definition 2.4]) and from the definition of orbital stability. We prove that \( \Gamma(u, \omega) \) is stable if condition (D) is satisfied. We argue by contradiction and suppose that there exists \( \varepsilon_{0} > 0 \) and \( (t_{n}, \Phi_{n})_{n \geq 1} \) such that
\[ d(\Phi_{n}, \Gamma(u, \omega)) \rightarrow 0, \quad d(U(t_{n}, \Phi_{n}), \Gamma(u, \omega)) \geq \varepsilon_{0}. \]
Thus, there exists \((u', \omega')\) in \( K_{C} \) such that
\[ \Gamma(u', \omega') \neq \Gamma(u, \omega) \]
and
\[ d(U(t_{n}, \Phi_{n}), \Gamma(u', \omega')) \rightarrow 0 \tag{7.19} \]
By Theorem 1.2, \( \mathbf{E}(U(t_{n}, \Phi_{n})) \rightarrow m_{C} \) and
\[ (\mathbf{P}(U(t_{n}, \Phi_{n})))_{n \geq 1} \]
is a minimising sequence of \( E \) over \( M_{C} \). By Theorem 1.1, up to extract a subsequence,
\[ \mathbf{P}(U(t_{n}, \Phi_{n})) \rightarrow \mathcal{G}(u'', \omega''). \tag{7.20} \]
for some \((u'', \omega'')\) in \( K_{C} \). By (7.19),
\[ \Gamma(u'', \omega'') = \Gamma(u', \omega'). \]
Now we set

\[ E_\delta := \inf_{\partial B_\delta} E > m_C. \]

The inequality follows from Theorem 1.1 and condition (D). For \( n \) large enough,

\[ E(U(t, \Phi_n)) = E(\Phi_n) < E_\delta \]

for every \( t \in \mathbb{R} \). By Proposition 5.1,

\[ E_\delta > E(P(U(t, \Phi_n))). \]

Our assumption on the regularity of the solutions of (CNLKG), ensures that \( U(\cdot, \Phi_n) \) is continuous in \( H^1 \). Then,

\[ P(U(t_n, \Phi_n)) \in B_\delta(G(u, \omega)) \]

otherwise the path \( P(U(\cdot, \Phi_n)) \) intersects the boundary of \( B \) where \( E \geq E_\delta \). By (7.20), \( G(u', \omega') \cap B_\delta \neq \emptyset \), so contradicting (D).

**References**


