DECAY OF EXCESS FOR THE ABELIAN HIGGS MODEL

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Abstract. In this article we prove that entire critical points (u,∇) of the self-dual $U(1)$-Yang–Mills–Higgs functional E_1, with normalized energy

$$\frac{E_1(u,\nabla;B_R)}{2\pi\omega_{n-2}R^{n-2}} := \frac{\int_{B_R}\left[|\nabla u|^2 + \frac{(1-|u|^2)^2}{4} + |F_\nabla|^2\right]}{2\pi\omega_{n-2}R^{n-2}} \leq 1+\tau(n)$$

for all $R>0$, have unique blow-down. Moreover, we show that they are two-dimensional in ambient dimension $2\leq n\leq 4$, or in any dimension $n\geq 2$ assuming that (u,∇) is a local minimizer, thus establishing a co-dimension-two analogue of Savin's theorem. The main ingredient is an Allard-type improvement of flatness.

Contents

1. Introduction

1.1. **Background on the Allen–Cahn and abelian Higgs models.** Area of geometric shapes is one of the oldest geometric functionals considered in mathematics. Given an ambient Riemannian manifold (M^n, g) (possibly the flat Euclidean space $\mathbb{R}^n$) and given an integer $1 \le k \le n-1$, one looks for k-dimensional objects, such as k-dimensional submanifolds or singular versions of them, which are *critical points* for the k-area $\mathcal{H}^k$. These are called *minimal submanifolds* (provided they are regular enough, depending on the context). Besides its intrinsic interest, the study of minimal submanifolds in a given ambient often reveals global topological structure, especially when coupled with curvature information.

These applications motivate a systematic existence and regularity theory of such critical points. In spite of its apparent simplicity, it is notoriously difficult to use the area functional directly in the context of the *calculus of variations*, especially when $k \ge 2$. Leaving out a number of very important ways to deal with this problem, such as the approach via parametrizations when $k = 2$ (see, e.g., [23, 45, 47, 46] among others), area-minimizing *currents* and *sets of finite perimeter* in the context of minimization (see [17, 25] and the monographs [26, 50]), and the Almgren–Pitts theory involving *varifolds* (see [2, 44]), in this paper we focus on the approximation of minimal submanifolds as limits of *diffuse* physical energies.

Starting from the pioneering ideas of De Giorgi, Modica [39], Ilmanen [34], and Hutchinson–Tonegawa [33], it was understood that smooth critical points

$u : M \to \mathbb{R}$ for the *Allen–Cahn energy*

$$E_\varepsilon(u) := \int_M \left[\varepsilon|du|^2 + \frac{(1-u^2)^2}{4\varepsilon}\right]$$

are effective diffuse approximations of minimal hypersurfaces. The Allen–Cahn functional is a well studied model for phase transitions; a typical critical point u takes values in $[-1,1]$, with $u \approx \pm 1$ (the pure phases) except in a transition region of thickness $\approx \varepsilon$, where most of the energy concentrates. Roughly speaking, this region is an ε-neighborhood of a minimal hypersurface, which acts as an interface between the two phases, and the energy density decays exponentially fast away from this interface.

This understanding brought a novel, PDE-based way to attack variational problems for the co-dimension-one area [29], which often allows to obtain more refined results compared to other methods [14].

In co-dimension two, similar attempts have been made by looking at the same energy for maps $u : M \to \mathbb{C}$, replacing u with $|u|$ in the second term. This corresponds to a simplified version of the Ginzburg–Landau model of superconductivity, popularized by Bethuel–Brezis–Hélein [8], where one neglects the magnetic field. The asymptotic analysis of this energy is substantially more involved, due to the lack of the aforementioned exponential decay, and brought mixed results: see, for instance, [38, 9] in the positive direction and [43] in the negative one.

On the other hand, including the magnetic field and looking at the so-called *self-dual regime* (also called *critical coupling*), we can consider the alternative energy

$$E_\varepsilon(u,\alpha) := \int_M \left[|du - i\alpha u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} + \varepsilon^2|d\alpha|^2\right].$$

Apart from the different normalization, it differs from the previous energies by an additional variable, the one-form $\alpha \in \Omega^1(M;\mathbb{R})$, which twists the Dirichlet term and appears in the Yang–Mills term $|d\alpha|^2$ (indeed, the latter equals $|F_\nabla|^2$, where F_∇ is the curvature of the connection $\nabla := d - i\alpha$ on the trivial complex line bundle $\mathbb{C} \times M$).

This energy, especially in this specific self-dual regime (i.e., the choice of constants in front of each term), is well known in gauge theory, where it is often called $U(1)$-*Yang–Mills–Higgs*, or simply *abelian Higgs model*. It received a thorough treatment in dimension 2, with a complete classification of critical planar pairs (u,∇) of finite energy by Taubes [51, 52]. See also [32] for the case of Riemann surfaces and [12] for Kähler manifolds. Recently, in [42], Stern and the third-named author developed the asymptotic analysis in arbitrary Riemannian manifolds, obtaining the precise co-dimension-two analogue of the result by Hutchinson–Tonegawa: see Theorem 4.1 below. Related facts, including Γ-convergence and the gradient flow convergence to mean curvature flow, have also been verified, by Parise, Stern, and the third-named author [40, 41].

Based on some new functional inequalities [30], the second-named author recently obtained a quantitative refinement of the work of Taubes, who showed (among other facts) that critical pairs on the plane minimize the energy among pairs with the same degree at infinity: namely, in [31] a quantitative *stability* is proved; the precise statement is recalled in Theorem 4.7. Together with the main result from [42], this result will be instrumental for the analysis in the present paper.

1.2. **Savin's theorem.** Since the work of De Giorgi [18] and Allard [2], it is known that almost-flat minimal submanifolds enjoy an *improvement of flatness*, i.e., they become even closer to a plane at smaller scales, in a quantitative way. Iteration of this improvement of flatness is the key mechanism in proving (quantitative) regularity of minimal submanifolds. The key analytical fact behind this decay property is the observation that the linearization of the minimal graph equation is the Laplace equation, whose solutions enjoy similar decay properties.

A related question, in the spirit of the classical Liouville theorem, is whether globally defined objects should be planar. The famous *Bernstein's conjecture* predicts that this is always true for minimal graphs $\mathbb{R}^{n-1} \to \mathbb{R}$, which are automatically (locally) area-minimizing hypersurfaces. In view of the improvement of flatness, this question quickly reduces to understanding whether any blow-down is necessarily a hyperplane. Bernstein's question was answered affirmatively by the works of Fleming, De Giorgi, Almgren, and Simons for $n \leq 8$, while Bombieri–De Giorgi–Giusti produced a counterexample for $n = 9$, whose blow-down corresponds to the Simons cone, in [11].

By analogy, De Giorgi conjectured that critical points $u : \mathbb{R}^n \to \mathbb{R}$ of the Allen–Cahn energy with $\frac{\partial u}{\partial x_n} > 0$ (so that level sets are graphs) are just rotations of a one-dimensional solution $u = u(x_n)$, at least when $n \leq 8$. The question has been solved by Ghoussoub–Gui for $n = 2$, in [28], by Ambrosio–Cabré for $n = 3$, in [4], and by Barlow–Bass–Gui under additional regularity assumptions for the level sets, in [6]. Finally, in [49] Savin settled the conjecture for all $n \leq 8$ under the assumption that $u(x', x_n) \to \pm 1$ as $x_n \to \pm\infty$, for any fixed $x' \in \mathbb{R}^{n-1}$. In fact, his main contribution could be phrased as follows.

Theorem 1.1 (Savin's theorem)**.** *A local minimizer u for Allen–Cahn enjoys improvement of flatness. In particular, if any blow-down is a hyperplane, then the blow-down is unique.*

Here the blow-downs can be understood in terms of energy concentration, or by looking at the blow-downs of the zero set $\{u = 0\}$ with respect to the (local) Hausdorff convergence of sets.

The previous statement implies the resolution of De Giorgi's conjecture for $n \leq 8$, with the extra assumption mentioned above. Indeed, it is known that this condition, together with $\frac{\partial u}{\partial x_n} > 0$, implies that u is a local minimizer; moreover, any blow-down gives a vertical area-minimizing cone in $\mathbb{R}^n$, hence an area-minimizing cone in $\mathbb{R}^{n-1}$, which is known to be necessarily a hyperplane for $n \le 8$.

In other words, uniqueness of the blow-down relies on two ingredients: improvement of flatness and a classification of blow-downs. While the second one can be directly exported from the setting of minimal hypersurfaces, the first one needs to be proved *before* passing to the limit $\varepsilon \to 0$, and this is the difficult part settled by Savin.

Finally, using the maximum principle (see, e.g., [24, 7]), one can deduce the following.

Corollary 1.2. *Under the previous assumptions, u is one-dimensional.*

As for minimal graphs, De Giorgi's conjecture (even with the extra assumption used by Savin) is false for $n \ge 9$: a counterexample has been constructed by Del Pino–Kowalczyk–Wei, in [22].

Savin's approach uses viscosity techniques, resembling the Krylov–Safanov theory in spirit. In particular, while his groundbreaking methods have a wide range of applicability, even beyond variational equations, it is not always clear how one can extend these techniques to the vectorial setting, where the maximum principle does not apply; see however [48, 21].

Recently, Wang [54] obtained a variational proof of Savin's theorem, following the strategy of Allard's proof of excess decay for stationary varifolds. Wang's paper has been the starting point for our investigation of the regularity properties of the zero set of solutions of the Yang–Mills–Higgs equations.

1.3. Main results.

We consider the energy

$$E_\varepsilon(u,\nabla) := \int e_\varepsilon(u,\nabla), \quad e_\varepsilon(u,\nabla) := |\nabla u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} + \varepsilon^2 |F_\nabla|^2.$$

Note that E_ε is just a rescaling of E_1, for $\varepsilon > 0$. The main result of the paper could be summarized as follows.

Theorem 1.3. *Savin's result, as stated in Theorem 1.1, holds for critical pairs (u,∇) for E_1, in any dimension $n \ge 2$.*

The following is the precise statement of the excess decay for critical points.

Theorem 1.4 (Tilt-excess decay)**.** *For any $n \ge 3$ and small enough $0 < \rho \le \rho_0(n)$, there exist constants $\varepsilon_0(n,\rho), \tau_0(n,\rho)$ such that the following holds. Let (u,∇) be a critical point for E_ε on the unit ball $B_1^n \subset \mathbb{R}^n$, with $\varepsilon \le \varepsilon_0$, $u(0) = 0$, and the energy bound*

$$\frac{1}{|B_1^{n-2}|} \int_{B_1^n} e_\varepsilon(u,\nabla) \le 2\pi + \tau_0. \tag{1.1}$$

Then at least one of the following statements is true: either

$$\mathbf{E}_1(u,\nabla,B_\rho^n,\overline{S}) \le C\rho^2 \mathbf{E}_1(u,\nabla,B_1^n,S),$$

for some $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \le C\sqrt{\mathbf{E}_1(u,\nabla,B_1^n,S)}$, where P_S is the orthogonal projection onto S, the plane minimizing $\mathbf{E}(u,\nabla,B_1^n,\cdot)$, or

$$\mathbf{E}_1(u,\nabla,B_1^n,S) \le \max\{C\varepsilon^2|\log\mathbf{E}|^2\sqrt{\mathbf{E}}, e^{-K/\varepsilon}\},$$

where $\mathbf{E} = \mathbf{E}(u,\nabla,B_1^n,S)$ and $C = C(n)$, $K = K(n)$ are independent of ρ.

Remark 1.5. To be precise, we assume also the pointwise bounds (4.1)–(4.2), which are automatically true if (u,∇) is a critical pair on $\mathbb{R}^n$ with energy growth $O(R^{n-2})$ on B_R^n.

Here $\mathbf{E}$ is the *excess*, defined in (5.1) below, which naturally splits into two parts, $\mathbf{E}_1$ and $\mathbf{E}_2$, measuring how far a solution is from being two-dimensional and from solving the first order *vortex equations*, respectively. We also note that $\mathbf{E}_1$ parallels the notion of excess in the theory of varifolds and does not depend on the orientation, while $\mathbf{E}$ sees the orientation and should be thought of as the stronger notion of excess in the setting of currents. While in principle the previous result establishes a quantitative decay only for $\mathbf{E}_1$, it is enough to obtain the following.

Corollary 1.6. *If (u,∇) is an entire critical point on $\mathbb{R}^n$, with*

$$0 < \lim_{R\to\infty} \frac{1}{|B_R^{n-2}|}\int_{B_R^n} e_\varepsilon(u,\nabla) \le 2\pi + \tau_0(n),$$

then this limit is 2π and the blow-down is a unique plane.

The previous limit always exists by the monotonicity formula for E_ε (see [42]). By a simple compactness argument and Allard's theorem, it is easy to see that the assumption guarantees that any blow-down is an $(n-2)$-dimensional plane. The key assertion is that, in view of improvement of flatness, the blow-down is *unique*.

Another simple consequence of the techniques is the following fact, a diffuse version of the $C^{1,\alpha}$ regularity of minimal graphs.

Theorem 1.7. *Let (u,∇) be a critical point for E_ε as above. Given $\alpha \in [0,1)$ and $\gamma > 0$, if $\varepsilon \le \varepsilon_0(n,\alpha,\gamma)$ and $\tau_0 \le \tau_0(n,\alpha,\gamma)$ then the vorticity set $\{|u| \le \frac{3}{4}\} \cap B_{1/2}^n$ is contained in a $C(n,\alpha,\gamma)\varepsilon^{1/(1+\alpha)}$-neighborhood of the graph of a function*

$$f : B_1^{n-2} \to \mathbb{R}^2$$

with $\|f\|_{C^{1,\alpha}} \le \gamma$.

Differently from the co-dimension one setting, where uniqueness of the blow-down (with multiplicity one) implies via the maximum principle that u is one-dimensional, at the present time we are not able to conclude that, in the setting of Corollary 1.6, the solution u is two-dimensional. Here we formulate the following variant of the *Gibbons conjecture*.

Conjecture 1.8. *An entire critical point (u,∇) on $\mathbb{R}^n$ satisfying*

$$\lim_{R\to\infty} \frac{1}{|B_R^{n-2}|}\int_{B_R^n} e_\varepsilon(u,\nabla) = 2\pi$$

and, writing any $x \in \mathbb{R}^n$ as $x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, also

$$\lim_{|y|\to\infty} |u(y, z)| = 1, \quad \text{uniformly in } z,$$

is necessarily two-dimensional, i.e., it is the pullback through the projection $\mathbb{R}^n \to \mathbb{R}^2$ of the standard solution in $\mathbb{R}^2$ with degree ± 1, up to translation and change of gauge.

It is interesting to note that, if we allow for multiplicity higher than one in the blow-down, this conjecture (with the appropriate energy assumption) does *not* hold for the non-magnetic Ginzburg–Landau energy mentioned before [16]. It is not clear if such rigidity with higher multiplicity should be expected for the energy considered in the present work. On the other hand, our excess decay is strong enough to give an affirmative answer up to dimension 4. With a more involved argument, we are able to settle it also for local minimizers in all dimensions $n \geq 2$, thus obtaining a full analogue of Savin's theorem.

Theorem 1.9. *The previous conjecture holds for critical points in dimension $2 \leq n \leq 4$, as well as for local minimizers in all dimensions $n \geq 2$, even without the second assumption that $\lim_{|y|\to\infty} |u(y, z)| = 1$ uniformly in z: the pair (u, ∇) is two-dimensional, up to rotation and change of gauge.*

The techniques used in this paper resemble those of [54] at several places. However, there are several key differences which require substantially new ideas. For instance, in order to construct the Lipschitz approximation, Wang uses a generic level set of u. The fact that typical level sets effectively share properties of minimal hypersurfaces is often used in [54], as well as in [34, 53, 14] and many other works in the Allen–Cahn setting. For the abelian Higgs model, level sets of u can be arbitrarily irregular, due to gauge invariance; while we can always pass to a local Coulomb gauge, we do not expect such effective properties of typical preimages of u.

Rather, in the present setting, we rely on the results from [31] in order to control in a fine way the behavior of u on many (but not all) two-dimensional slices perpendicular to the reference plane. For instance, we are able to bound the distance of the actual zero set from a certain graph giving the "center of mass" of each slice, which is used as a Lipschitz approximation and allows to derive a Caccioppoli-type inequality.

In the case of minimizers, this refined control also allows us to deform a nearly flat minimizing pair (u, ∇) in the interior to gain a *stronger* decay of the excess. This deformation process also requires a very involved covering argument and gauge fixing procedure.The following theorem is the precise statement of the improved tilt-excess decay that we obtain for minimizers. Note that, in the statement below, β can indeed be any power. This is fundamental for proving Theorem 1.9 for minimizers, where we need to take β large enough.

Theorem 1.10. *For any $\beta > 0$ and small enough $0 < \rho \leq \rho_0(n, \beta)$ there exist $\tau_0(n, \beta, \rho) > 0$, $\varepsilon_0(n, \beta, \rho) > 0$ with the following property. Let (u, ∇) be a* *local minimizer of E_ε in B_1^n with $\varepsilon \le \varepsilon_0$ and $u(0) = 0$ such that*

$$\frac{1}{|B_1^{n-2}|}\int_{B_1^n} e_\varepsilon(u,\nabla) \le 2\pi + \tau_0,$$

and let S minimize $\mathbf{E}(u,\nabla,B_1^n,S)$. Then, after a suitable rotation, at least one of the following statements is true: either

$$\mathbf{E}(u,\nabla,B_\rho^n,\overline{S}) \le C\rho^2\mathbf{E}(u,\nabla,B_1^n,S),$$

for some new oriented $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \le C\sqrt{\mathbf{E}}$, or

$$\mathbf{E}(u,\nabla,B_1^n,\mathbb{R}^{n-2}) \le \varepsilon^\beta,$$

where $C = C(n,\beta)$ is independent of ρ.

Then, by taking $\beta > n-2$, we obtain a direct proof of Theorem 1.9 in the case of minimizers.

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2. Basic definitions

While we work on the trivial Hermitian line bundle over the Euclidean space $\mathbb{R}^n$, it is worth to recall the definition of Hermitian line bundle over a general manifold.

Definition 2.1. A *Hermitian line bundle* over a smooth manifold M is a complex line bundle $L \to M$ (i.e., a complex vector bundle with typical fiber $\mathbb{C}$) equipped with a *Hermitian metric*, whose real part will be denoted by $\langle\cdot,\cdot\rangle$; thus, for any two smooth sections $s,t \in \Gamma(L)$, the function $p \mapsto \langle s(p),t(p)\rangle$ is smooth and real-valued, and satisfies $\langle is(p),it(p)\rangle = \langle s(p),t(p)\rangle = \langle t(p),s(p)\rangle$.

Definition 2.2. A *metric connection* is a map ∇ which assigns to each vector field $\xi \in \Gamma(TM)$ an endomorphism $\nabla_\xi : \Gamma(L) \to \Gamma(L)$ with the following properties:

(i) $\nabla_{\xi+\eta}s = \nabla_\xi s + \nabla_\eta s$;
(ii) $\nabla_{\phi\xi}s = \phi\nabla_\xi s$;
(iii) $\nabla_\xi(\phi s) = (\xi\phi)s + \phi\nabla_\xi s$;
(iv) $\xi(\langle s,t\rangle) = \langle\nabla_\xi s,t\rangle + \langle s,\nabla_\xi t\rangle$,

for any sections $s,t \in \Gamma(L)$, vector fields $\xi,\eta \in \Gamma(TM)$, and function $\phi \in C^\infty(M)$.

On the trivial bundle $L = \mathbb{C}\times M$, we can always write a metric connection ∇ as

$$\nabla = d - i\alpha,$$

for a real-valued one-form, meaning that $\nabla_\xi s = ds(\xi) - i\alpha(\xi)s$.

In general, for two vector fields ξ and η, typically ∇_ξ and ∇_η do not commute, meaning that the connection has nontrivial *curvature*. Formally, the curvature F_∇ is given by

$$F_\nabla(\xi,\eta)(s) = [\nabla_\xi, \nabla_\eta]s - \nabla_{[\xi,\eta]}s. \tag{2.1}$$

A simple computation shows that F_∇ is a two-form with values in the Lie algebra of $U(1)$, i.e., in imaginary numbers; we will sometimes use the real-valued two-form ω given by

$$F_\nabla(\xi,\eta)(s) =: -i\omega(\xi,\eta)s. \tag{2.2}$$

On the trivial bundle, if $\nabla = d - i\alpha$ then we simply have

$$\omega = d\alpha.$$

We will use the inner product on two-forms induced by the following quadratic form:

$$|\omega|^2 = \sum_{1\le j<k\le n} |\omega(e_j, e_k)|^2,$$

where $\{e_k\}_{k=1}^n$ is a local orthonormal frame for TM.

3. The abelian Higgs equations

For a section $u \in \Gamma(L)$ and a (metric) connection ∇ on a Hermitian line bundle $L \to M$ over a smooth Riemannian manifold (M, g), given a parameter $\varepsilon > 0$, we define the $U(1)$-Yang–Mills–Higgs energy as

$$E_\varepsilon(u,\nabla) := \int_M \left[|\nabla u|^2 + \varepsilon^2|F_\nabla|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}\right], \tag{3.1}$$

where F_∇ is the curvature of ∇ and $|F_\nabla|$ is defined to be $|\omega|$ (with ω as in (2.2)). Equivalently, on the trivial bundle, for any section u (viewed as a function $M \to \mathbb{C}$) and connection $\nabla = d - i\alpha$ we have

$$E_\varepsilon(u, \nabla = d - i\alpha) = \int_M \left[|du - iu\alpha|^2 + \varepsilon^2|d\alpha|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2}\right].$$

A smooth pair (u, ∇) gives a critical point for the Yang–Mills–Higgs energy if and only if it satisfies the system of partial differential equations:

$$\nabla^*\nabla u = \frac{1}{2\varepsilon^2}(1-|u|^2)u, \tag{3.2}$$

$$\varepsilon^2 d^*\omega = \langle \nabla u, iu\rangle, \tag{3.3}$$

where ∇^* is the adjoint of ∇, while d^* is the adjoint of $d : \Omega^1(M) \to \Omega^2(M)$, given by

$$(d^*\omega)(e_k) = -\sum_{j=1}^n (\nabla_{e_j}\omega)(e_j, e_k)$$

for some (and hence any) orthonormal frame $\{e_j\}$.

We now recall some Bochner-type identities from [42, Sections 2–3]. Since ω is a closed two-form, after taking the exterior derivative in (3.3) we get

$$\varepsilon^2\Delta_H\omega + |u|^2\omega = \psi(u), \tag{3.4}$$

where $\Delta_H = dd^* + d^*d$ is the Hodge Laplacian and

$$\psi(u)(e_j, e_k) := 2\langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle. \tag{3.5}$$

One easily sees that the modulus $|u|^2$ satisfies the equation

$$\Delta\frac{1}{2}|u|^2 = |\nabla u|^2 - \frac{|u|^2}{2\varepsilon^2}(1-|u|^2). \tag{3.6}$$

We also recall the following Bochner identity for $|\nabla u|^2$:

$$\Delta\frac{1}{2}|\nabla u|^2 = |\nabla^2 u|^2 + \frac{1}{2\varepsilon^2}(3|u|^2-1)|\nabla u|^2 - 2\langle\omega, \psi(u)\rangle + \mathcal{R}_1(\nabla u, \nabla u), \tag{3.7}$$

where $\mathcal{R}_1 = \mathrm{Ric}(e_j, e_k)\langle\nabla_{e_j}u, \nabla_{e_k}u\rangle$ and $\nabla^2_{e_j,e_k}u = \nabla_{e_j}(\nabla_{e_k}u)$.

Next, we define the gauge-invariant Jacobian, which plays an important role in the Γ-convergence theory [40], similar to the classical Jacobian in the Γ-convergence for the Ginzburg–Landau energy with no magnetic field [1, 9, 37]. It is the two-form given by

$$J(u,\nabla) := \psi(u) + (1-|u|^2)\omega. \tag{3.8}$$

We have the trivial pointwise bound

$$|J(u,\nabla)| \le e_\varepsilon(u,\nabla), \tag{3.9}$$

where $e_\varepsilon(u,\nabla)$ is the integrand in (3.1).

We define Γ_ε to be the dual current to the Jacobian, formally identified by the duality formula

$$\langle\Gamma_\varepsilon, \xi\rangle = \frac{1}{2\pi}\int_M J(u,\nabla)\wedge\xi, \tag{3.10}$$

for any $(n-2)$-form $\xi \in \Omega^{n-2}(M)$. Note that by (3.4) the Jacobian can be written as

$$J(u,\nabla) = \omega + \varepsilon^2\Delta_H\omega.$$

In particular, this shows that the gauge-invariant Jacobian is a closed two-form. This, in duality, implies that $\partial\Gamma_\varepsilon = 0$, i.e., Γ_ε is an $(n-2)$-dimensional cycle.

4. Preliminary estimates

4.1. **The energy concentration set.** It was proved by the third-named author and Stern in [42] that, for a sequence $(u_\varepsilon, \nabla_\varepsilon)$ with $\varepsilon \to 0$, one can extract a subsequence such that the energy density converges to (the weight of) a stationary integer-rectifiable $(n-2)$-varifold. We restate the main result of [42] in the following theorem (see [42, eq. (6.35)] for the conclusion on the Jacobian).

Theorem 4.1 (The varifold limit)**.** *Let $L \to M$ be a Hermitian line bundle over a closed, oriented Riemannian manifold (M^n, g) of dimension $n \geq 2$ and let $(u_\varepsilon, \nabla_\varepsilon)$ be a family of critical points of E_ε, satisfying the uniform energy bound*

$$E_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \leq \Lambda < \infty.$$

Then, as $\varepsilon \to 0$, the energy measures

$$\mu_\varepsilon = \frac{1}{2\pi} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\,\mathrm{vol}_g,$$

converge subsequentially, in duality with $C^0(M)$, to a measure μ which is the weight of a stationary integral $(n-2)$-varifold V. Also, for all $0 \leq \delta < 1$,

$$\mathrm{spt}(V) = \lim_{\varepsilon\to 0}\{|u_\varepsilon| \leq \delta\},$$

in the Hausdorff topology. The $(n-2)$-currents dual to the curvature forms $\frac{1}{2\pi}\omega_\varepsilon$ and Jacobians $\frac{1}{2\pi}J(u_\varepsilon, \nabla_\varepsilon)$ converge subsequentially to the same limit, an integral cycle Γ with $|\Gamma| \leq \mu$.

Remark 4.2. The previous result admits a local version, proved in the same way (assuming the bounds (4.1) and (4.2) below, which in the closed case follow from the maximum principle): assume that we have an increasing sequence of open sets $U_\varepsilon \subseteq \mathbb{R}^n$ and a sequence of smooth pairs $(u_\varepsilon, \nabla_\varepsilon)$, each defined on the trivial bundle $\mathbb{C} \times U_\varepsilon$ and critical for E_ε; if we have

$$\limsup_{\varepsilon\to 0} \int_K e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) < \infty$$

for any compact subset $K \subset U := \bigcup_\varepsilon U_\varepsilon$, as well as (4.1)–(4.2), then there exist a limiting varifold V and a limiting cycle Γ satisfying the same conclusions as above (up to a subsequence).

We will use the above theorem (in its local version) in several soft arguments by compactness and contradiction; in particular, we will use it to obtain information for any blow-down limit of an entire solution.

4.2. **Modica-type bounds and exponential decay.** Actually, [42] contains some additional information which will be used frequently in the paper, including a Modica-type bound which was first proved in dimension two in [36, Theorem III.8.1]. We record the following propositions in the non-compact case of $M = \mathbb{R}^n$, with the trivial bundle $L = \mathbb{C} \times \mathbb{R}^n$.

Proposition 4.3. *A critical point (u, ∇) for E_ε, on the trivial bundle on $\mathbb{R}^n$, satisfies*

$$|u| \leq 1 \tag{4.1}$$

everywhere.

Proof. The bound $|u| \leq 1 + C(n)\varepsilon^2$ on the unit ball $B_1(0)$ can be shown as in [41, Proposition A.2], and the claim follows by scaling. □

Proposition 4.4 (Modica-type bounds)**.** *Assuming also that the energy on a ball B_R is $O(R^{n-2})$ for R large enough, we have the pointwise bounds*

$$\varepsilon|F_\nabla| \le \frac{1-|u|^2}{2\varepsilon}, \quad |\nabla u| \le \frac{1-|u|^2}{\varepsilon}. \tag{4.2}$$

Proof. The proof is essentially the same as in [42]; however, in the Euclidean space, the Modica-type bound has no error terms. First, define ξ_ε to be the *discrepancy*:

$$\xi := \varepsilon|F_\nabla| - \frac{1-|u|^2}{2\varepsilon}. \tag{4.3}$$

Arguing as in [42, Section 3], we see that

$$\Delta\xi \ge \frac{|u|^2}{\varepsilon^2}\xi. \tag{4.4}$$

For the positive part ξ^+, this immediately implies that

$$\Delta\xi^+ \ge 0$$

in the distributional sense, i.e., ξ is subharmonic. Under the energy growth assumption, we have

$$\int_{B_R(0)} |\xi| = O(R^{n-1}),$$

which gives $\xi^+ \equiv 0$, as claimed.

For the second bound, proceeding as in [42, eqs. (5.5)–(5.6)], we check that

$$w := |\nabla u| - \frac{1-|u|^2}{\varepsilon}$$

satisfies

$$\Delta w \ge \frac{|u|^2}{\varepsilon^2}w + \frac{1}{\varepsilon}\left(w + \frac{1-|u|^2}{\varepsilon}\right)\left(2w + \frac{1-|u|^2}{2\varepsilon}\right).$$

Again, this implies that w^+ is subharmonic, and hence $w^+ \equiv 0$. □

We also record the following exponential decay of energy, which plays a key role in the paper.

Proposition 4.5 (Exponential decay away from the vorticity set)**.** *For critical pairs (u, ∇) with bounds (4.1) and (4.2) there exist constants $K(n) > 0$ and $C(n) > 0$ such that, defining $Z := \{|u| \le \frac{3}{4}\}$ and $r(p) := \operatorname{dist}(p, Z)$, we have*

$$e_\varepsilon(u, \nabla) \le C\frac{e^{-Kr(p)/\varepsilon}}{\varepsilon^2}. \tag{4.5}$$

Proof. As in [42, Corollary 5.2], we compute that on $\mathbb{R}^n \setminus Z$ we have

$$\Delta\frac{1-|u|^2}{2} \ge \frac{1-|u|^2}{4\varepsilon^2}.$$

Exponential decay now follows as in [42, Proposition 5.3], using also the previous Modica-type bounds. □

4.3. **Inner variations and monotonicity.** In this section we recall the inner variation formulas for critical points. With respect to any orthonormal basis $\{e_k\}_{k=1}^n$ for TM, we define the $(0,2)$-tensors $\nabla u^*\nabla u$ and $\omega^*\omega$ by

$$(\nabla u^*\nabla u)(e_j,e_k) := \langle \nabla_{e_j}u, \nabla_{e_k}u\rangle, \tag{4.6}$$

$$\omega^*\omega(e_i,e_j) := \sum_{k=1}^n \omega(e_i,e_k)\omega(e_j,e_k). \tag{4.7}$$

We define the *stress-energy tensor* to be

$$T_\varepsilon(u,\nabla) := e_\varepsilon(u,\nabla) - 2\nabla u^*\nabla u - 2\varepsilon^2\omega^*\omega. \tag{4.8}$$

Then, for any pair (u,∇) satisfying (3.2)–(3.3), the inner variation formula then reads

$$\operatorname{div}(T_\varepsilon(u,\nabla)) = 0, \tag{4.9}$$

meaning that, for any compactly supported vector field X,

$$\int_M \langle T_\varepsilon(u,\nabla), DX\rangle = 0. \tag{4.10}$$

A core tool in the proof of Theorem 4.1 is the *monotonicity formula* from [42, Theorem 4.3], which is cleaner in the case of the trivial line bundle $L = \mathbb{C}\times\mathbb{R}^n$ over the flat Euclidean space $M = \mathbb{R}^n$. We state this version of the theorem for convenience and give a short proof.

Proposition 4.6 (Monotonicity formula)**.** *Let (u,∇) be a critical point for E_ε on the trivial line bundle $L = \mathbb{C}\times\mathbb{R}^n \to \mathbb{R}^n$. Then the normalized energy*

$$\tilde{E}_\varepsilon(p,r) := r^{2-n}\int_{B_r(p)} e_\varepsilon(u,\nabla)$$

satisfies

$$\begin{aligned}\frac{d}{dr}\tilde{E}_\varepsilon(p,r) = {}& 2r^{1-n}\int_{B_r(p)}\left(\frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2|\omega|^2\right)\\ &+ 2r^{2-n}\int_{\partial B_r(p)}(|\nabla_\nu u|^2 + \varepsilon^2|\iota_\nu\omega|^2).\end{aligned} \tag{4.11}$$

Proof. Without loss of generality, assume that $p=0$. By approximation we can take $X(x) = \mathbf{1}_{B_r(0)}\sum_{k=1}^n x_k e_k$ in (4.10), obtaining

$$\begin{aligned} r\int_{\partial B_r} e_\varepsilon(u,\nabla) = {}& \int_{B_r}(n-2)e_\varepsilon(u,\nabla) + 2\int_{B_r}\left(\frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2|\omega|^2\right)\\ &+ 2r\int_{\partial B_r}\left(|\nabla_\nu u|^2 + \varepsilon^2|\iota_\nu\omega|^2\right).\end{aligned}$$

Since

$$\begin{aligned}\frac{d}{dr}\tilde{E}_\varepsilon(x,r) &= (2-n)r^{1-n}\int_{B_r} e_\varepsilon(u,\nabla) + r^{2-n}\int_{\partial B_r} e_\varepsilon(u,\nabla)\\ &= 2r^{1-n}\int_{B_r}\left(\frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2|\omega|^2\right)\\ &\quad + 2r^{2-n}\int_{\partial B_r}(|\nabla_\nu u|^2 + \varepsilon^2|\iota_\nu\omega|^2),\end{aligned}$$

we obtain the desired conclusion. □

4.4. Quantitative stability in two dimensions and the vortex equations.

In this section we record some results regarding the existence, uniqueness, and quantitative stability of critical points for (3.1) in $\mathbb{R}^2$. First of all note that for $\varepsilon = 1$ the energy E_1 of *any* pair (u,∇) can be written as follows:

$$\begin{aligned}E(u,\nabla) &= \int_{\mathbb{R}^2}\left[|\nabla u|^2 + |F_\nabla|^2 + \frac{(1-|u|^2)^2}{4}\right]\\ &= 2\pi|N| + \int_{\mathbb{R}^2}|\nabla_1 u \pm i\nabla_2 u|^2 + \left|\star\omega \mp \frac{1-|u|^2}{2}\right|^2,\end{aligned} \tag{4.12}$$

where N is the vortex number of (u,∇), given by

$$N := \frac{1}{2\pi}\int_{\mathbb{R}^2}\star\omega.$$

Thus (u,∇) is a minimizer of the total energy among pairs with the same vortex number if and only if it satisfies the first-order system of *vortex equations*:

$$\nabla_1 u \pm i\nabla_2 u = 0 \text{ and } \star\omega = \pm\frac{1-|u|^2}{2\varepsilon}. \tag{4.13}$$

These are also called *Bogomol'nyi equations* (after [10]) or *self-dual equations*, and arise in many self-dual gauge theories. Taubes, in [51], proved that we can prescribe the zero set $u^{-1}(0) = \{a_1,\dots,a_k\}$: given any finite collection of $k \geq 0$ points, counted with multiplicity, there exists a solution (u,∇) to the vortex equations (with either choice of signs, corresponding to vortex number $N = k$ and $N = -k$, respectively) with this prescribed zero set; moreover, the solution is unique up to change of gauge.

In [31] the second-named author improved the previous results by proving a (sharp) quantitative stability for critical points of E_1. We record these results and this improvement in the following theorem.

Theorem 4.7 (Uniqueness and stability in two dimensions)**.** *On the trivial line bundle over $\mathbb{R}^2$, any critical point (u,∇) of finite energy for E_1 is actually a minimizer with $E_1(u,\nabla) = 2\pi k \in 2\pi\mathbb{N}$. Moreover, up to change of gauge, any minimizer is uniquely characterized by its zero set $u^{-1}(0) = \{a_1,\dots,a_k\}$ (counted with multiplicity, according to the local degree of u around any zero)* *and orientation. Letting $\mathcal{F}$ be the moduli space of all minimizers, the following quantitative stability estimates hold:*

$$\inf_{(u_0,\nabla_0)\in\mathcal{F}}(\|u-u_0\|^2_{L^2(\mathbb{R}^2)}+\|F_\nabla-F_{\nabla_0}\|^2_{L^2(\mathbb{R}^2)})\le C_k(E_1(u,\nabla)-2\pi k), \tag{4.14}$$

for some constant $C_k>0$ and all pairs such that the discrepancy $E_1(u,\nabla)-2\pi k\le\delta_k$ is small enough.

Proof. Existence and uniqueness were proved in [51, 52], while quantitative stability was obtained in [31]. □

The proof of the above theorem uses weighted estimates developed in [30]. Essentially, Theorem 4.7 tells us that in the vanishing ε limit, two-dimensional slices perpendicular to the energy concentration set resemble minimizing vortex solutions in $\mathbb{R}^2$. In the case of regular enough pairs (u,∇), we also have the stability of the Jacobian and the energy density, given by the following theorem.

Theorem 4.8. *For any $\Lambda>1$ and integer k, there exist constants $C_{\Lambda,k}>0$ and $\eta_{\Lambda,k}>0$ with the following property. Let $(u,\nabla)\in W^{1,2}_{loc}(\mathbb{R}^2)$ be a finite-energy pair such that*

- *(i) $\Lambda^{-1}|u_0|\le|u|\le\Lambda|u_0|$ for a solution (u_0,∇_0) with zero set $\{x_j\}_{j=1}^k$ (counted with multiplicity);*
- *(ii) $E_1(u,\nabla)-2\pi k\le\eta^2_{\Lambda,k}$;*
- *(iii) $\frac{u}{|u|}\in W^{1,1}_{loc}$ and has the same degree as $\frac{u_0}{|u_0|}$ around each x_j.*

Then for any $0<\gamma<\frac{1}{k}$, writing $\nabla=d-i\alpha$, we have

$$\int_{\mathbb{R}^2}|u_0|^{2+2\gamma}\left[\left|d\log\left(\frac{|u|}{|u_0|}\right)\right|^2+|\alpha-\alpha_0|^2\right]\le\frac{C_{\Lambda,k}}{\gamma^2}[E_1(u,\nabla)-2\pi k],$$

up to a change of gauge. Moreover, the Jacobian and the energy density satisfy the following estimates:

$$\begin{aligned}&\|J(u,\nabla)-J(u_0,\nabla_0)\|_{L^1(\mathbb{R}^2)}+\|e_1(u,\nabla)-e_1(u_0,\nabla_0)\|_{L^1(\mathbb{R}^2)}\\&\le C_{\Lambda,k}\sqrt{E_1(u,\nabla)-2\pi k}.\end{aligned}\tag{4.15}$$

Proof. For the proof see [31, Theorems 1.2 and 1.3], as well as [31, Section 3.3]. □

5. Quantifying flatness and the excess

We assume that $n\ge3$ throughout the rest of the paper, unless otherwise stated.

5.1. **Excess definitions.** In this section we introduce a way to measure *flatness* of a pair (u,∇). Inspired by the definition of tilt-excess by De Giorgi [18], we define the *Yang–Mills–Higgs excess* as

$$\begin{aligned}\mathbf{E}(u,\nabla,B_r(x),S) &:= \frac{r^{2-n}}{2\pi}\int_{B_r(x)}[e_\varepsilon(u,\nabla)-J(u,\nabla)\wedge e_S^*] \\ &= \mu_\varepsilon(B_r(x))-\langle\Gamma_\varepsilon,\mathbf{1}_{B_r(x)}e_S^*\rangle\end{aligned}\tag{5.1}$$

for any *oriented* $(n-2)$-plane S in $\mathbb{R}^n$ with the associated $(n-2)$-vector e_S and $(n-2)$-covector e_S^*. Take an oriented orthonormal basis of $S=\operatorname{span}\{e_3,\dots,e_n\}$ and extend it to an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$. Then by a completion of squares we see that the excess splits into two terms:

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2,$$

where

$$\begin{aligned}&\mathbf{E}_1(u,\nabla,B_r(x),S) \\ &:= \frac{r^{2-n}}{2\pi}\int_{B_r(x)}\left[\sum_{k=3}^n|\nabla_{e_k}u|^2+\varepsilon^2\sum_{(j,k)\neq(1,2)}\omega(e_j,e_k)^2\right]\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}&\mathbf{E}_2(u,\nabla,B_r(x),S) \\ &:= \frac{r^{2-n}}{2\pi}\int_{B_r(x)}\left[|\nabla_{e_1}u+i\nabla_{e_2}u|^2+\left|\varepsilon\omega(e_1,e_2)-\frac{1-|u|^2}{2\varepsilon}\right|^2\right].\end{aligned}\tag{5.3}$$

Note that $\mathbf{E}_1$ quantifies how flat the solution is in the directions tangent to S, while $\mathbf{E}_2$ quantifies the error in the vortex equations on perpendicular slices. Moreover, $\mathbf{E}_1$ does *not* depend on the orientation of S (while $\mathbf{E}$ and $\mathbf{E}_2$ do).

The Yang–Mills–Higgs excess is a key tool in our analysis. For $S:=\{0\}\times\mathbb{R}^{n-2}$, with a slight abuse of notation, we define

$$\mathbf{E}_z=\frac{1}{2\pi}\int_{B_1^2\times\{z\}}[e_\varepsilon(u,\nabla)-J(u,\nabla)(e_1,e_2)]$$

for $z\in\mathbb{R}^{n-2}$, and similarly

$$\begin{aligned}(\mathbf{E}_1)_z &:= \frac{1}{2\pi}\int_{B_1^2\times\{z\}}\left[\sum_{k=3}^n|\nabla_{e_k}u|^2+\varepsilon^2\sum_{(j,k)\neq(1,2)}\omega(e_j,e_k)^2\right], \\ (\mathbf{E}_2)_z &:= \frac{1}{2\pi}\int_{B_1^2\times\{z\}}\left[|\nabla_{e_1}u+i\nabla_{e_2}u|^2+\left|\varepsilon\omega(e_1,e_2)-\frac{1-|u|^2}{2\varepsilon}\right|^2\right].\end{aligned}$$

5.2. **The tilt-excess decay statement.** Parallel to De Giorgi's [18] and Allard's [2] regularity theorems, we aim to prove a *decay of the excess* up to scale ε, compare with [54, Theorem 3.3]. More precisely, our goal is to show Theorem 1.4, which is one of the main results of the present work. For convenience, we recall its statement here.

Theorem 5.1. *For any $n \geq 3$ and small enough $0 < \rho \leq \rho_0(n)$ there exist constants $C(n) > 0$ and $\varepsilon_0(n, \rho), \tau_0(n, \rho)$ such that the following holds. Let (u, ∇) be a critical point for the energy E_ε, given by (3.1), with $\varepsilon \leq \varepsilon_0$. Assume that u satisfies the bounds (4.1) and (4.2), that $u(0) = 0$, and the energy bound*

$$\frac{1}{|B_1^{n-2}|} \int_{B_1^n} e_\varepsilon(u, \nabla) \leq 2\pi + \tau_0.$$

Then at least one of the following statements is true: either

$$\mathbf{E}_1(u, \nabla, B_\rho^n, \overline{S}) \leq C(n)\rho^2 \mathbf{E}_1(u, \nabla, B_1^n, S), \tag{5.4}$$

for some $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \leq C(n)\sqrt{\mathbf{E}_1(u, \nabla, B_1^n, S)}$, where P_S is the orthogonal projection onto S, the plane minimizing $\mathbf{E}(u, \nabla, B_1^n, \cdot)$, and $\|\cdot\|$ is the Hilbert–Schmidt norm, or

$$\mathbf{E}_1(u, \nabla, B_1^n, S) \leq \max\{C(n)\varepsilon^2 |\log \mathbf{E}|^2 \sqrt{\mathbf{E}}, e^{-K(n)/\varepsilon}\}, \tag{5.5}$$

where $\mathbf{E} = \mathbf{E}(u, \nabla, B_1^n, S)$.

Note that thanks to Proposition 4.3 and Proposition 4.4, if u is an entire solution such that $\int_{B_R^n} e_\varepsilon(u, \nabla) = O(R^{n-2})$, then (4.1) and (4.2) are statisfied. In particular by scaling we deduce the following.

Theorem 5.2. *For any small enough $0 < \rho \leq \rho_0(n)$, there exist constants $C(n), R_0(n, \rho) > 0$ and $\tau_0(n, \rho)$ with the following property. Let (u, ∇) be an entire critical point for E_1, with the energy bound*

$$\lim_{R \to \infty} \frac{1}{|B_R^{n-2}|} \int_{B_R^n} e_1(u, \nabla) \leq 2\pi + \tau_0.$$

Then for all $R \geq R_0$ at least one of the following statements is true: either

$$\mathbf{E}_1(u, \nabla, B_{\rho R}^n, \overline{S}) \leq C(n)\rho^2 \mathbf{E}_1(u, \nabla, B_R^n, S), \tag{5.6}$$

for some $(n-2)$-plane $\overline{S}$ with $\|P_{\overline{S}} - P_S\| \leq C(n)\sqrt{\mathbf{E}_1(u, \nabla, B_R^n, S)}$ and S minimizing $\mathbf{E}(u, \nabla, B_R^n, \cdot)$, or

$$\mathbf{E}_1(u, \nabla, B_R^n, S) \leq \max\{C(n)R^{-2} |\log \mathbf{E}|^2 \sqrt{\mathbf{E}}, e^{-K(n)R}\}, \tag{5.7}$$

where $\mathbf{E} = \mathbf{E}(u, \nabla, B_R^n, S)$.

5.3. **Blow-up at multiplicity one points.** Allard's regularity theorem [2] asserts that the energy concentration set in Theorem 4.1 is locally a $C^{1,\alpha}$ submanifold around points of multiplicity one. We use this to show that, for any blow-down, the energy concentration set is a flat $(n-2)$-plane.

Proposition 5.3 (Multiplicity one and vanishing of excess)**.** *For any $\delta > 0$ there exist $\tau_0(n, \delta) > 0$ and $\varepsilon_0(n, \delta) > 0$ small enough with the following property. Let (u, ∇) be a critical point for E_ε on the unit ball B_1^n, with $u(0) = 0$ and $\varepsilon \leq \varepsilon_0$, as well as the energy bound*

$$\frac{1}{|B_1^{n-2}|} \int_{B_1^n} e_\varepsilon(u, \nabla) \leq 2\pi + \tau_0$$

and (4.1)–(4.2). Then, after a suitable rotation and, possibly, a conjugation of (u, ∇),

$$\mathbf{E}(u, \nabla, B^n_{1/2}, \mathbb{R}^{n-2}) \le \delta,$$

where we write $\mathbb{R}^{n-2}$ to mean $\{0\} \times \mathbb{R}^{n-2}$. As a consequence, given an entire critical point $(\tilde{u}, \tilde{\nabla})$ for E_1, with $\tilde{u}(0) = 0$ and the energy bound

$$\lim_{R\to\infty} \frac{1}{|B^{n-2}_R|} \int_{B^n_R} e_1(\tilde{u}, \tilde{\nabla}) \le 2\pi + \tau_0(n),$$

then the previous limit is 2π and we can find oriented $(n-2)$-planes $S(R)$ such that

$$\lim_{R\to\infty} \mathbf{E}(\tilde{u}, \tilde{\nabla}, B^n_R, S(R)) = 0.$$

Proof. The proof is a standard argument by compactness and contradiction.

Local case. Assume that there are sequences $(u_\varepsilon, \nabla_\varepsilon)$ and $\tau_\varepsilon \to 0$ (as $\varepsilon \to 0$) such that

$$\int_{B^n_1} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \le (2\pi + \tau_\varepsilon)|B^{n-2}_1|$$

and, on the other hand,

$$\liminf_{\varepsilon\to 0} \mathbf{E}(u_\varepsilon, \nabla_\varepsilon, B^n_{1/2}, S(\varepsilon)) > 0, \tag{5.8}$$

for any choice of oriented $(n-2)$-planes $S(\varepsilon)$ (where, with abuse of notation, we write ε to mean a sequence $\varepsilon_k \to 0$). We apply Theorem 4.1: up to extracting a subsequence, we have

$$e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\, dx \overset{*}{\rightharpoonup} 2\pi\, d\mu_V$$

in duality with C^0_c, where V is a stationary integral $(n-2)$-varifold whose weight μ_V obeys the bound

$$\mu_V(B^n_1) \le \liminf_{\varepsilon\to 0} \frac{1}{2\pi} \int_{B^n_1} e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) \le |B^{n-2}_1|. \tag{5.9}$$

Moreover there exists an integral $(n-2)$-cycle Γ such that $J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup 2\pi\Gamma$ as currents and $|\Gamma| \le \mu_V$.

Since $u_\varepsilon(0) = 0$, by the clearing-out lemma [42, Corollary 4.4] we get that $0 \in \operatorname{spt}(\mu_V)$, so that $\Theta^{n-2}(\mu_V, 0) \ge 1$ since V is an integral stationary varifold. Because of (5.9), the monotonicity formula for stationary varifolds is saturated, showing that V must be a cone with respect to the origin; we extend it to a stationary cone $\tilde{V}$ on $\mathbb{R}^n$. Since $\Theta^{n-2}(\mu_{\tilde{V}}, x) \ge 1 = \Theta^{n-2}(\mu_{\tilde{V}}, 0)$ for all $x \in \operatorname{spt}(\mu_{\tilde{V}})$, we see that $\tilde{V}$ is a cone with respect to any $x \in \operatorname{spt}(\mu_{\tilde{V}})$, and hence a plane (since the tangent plane exists for a.e. point x). Thus, up to a rotation, V is the multiplicity-one varifold associated to $\{0\} \times \mathbb{R}^{n-2}$.

Moreover, the argument used in [42, Section 6.2] to show integrality of V actually reveals that the limiting density is the sum of the absolute values of the degrees of

$$\frac{u_\varepsilon}{|u_\varepsilon|}\Big|_{\partial D_i \times \{z\}}$$

along typical slices $B_1^2 \times \{z\}$ with $|z| < \frac{1}{2}$, where $D_1, \dots, D_N \subset B_{1/2}^2$ are suitable disjoint disks (depending on z) such that $u_\varepsilon(\cdot, z) \neq 0$ on $B_{1/2}^2 \setminus \bigcup_i D_i$ (see in particular the proof of [42, Proposition 6.6] and the conclusion of [42, Proposition 6.7]). Since the limiting density is 1 and, eventually, $u_\varepsilon(y, z) \neq 0$ for $y \in \partial B_{1/2}^2$ and $z \in B_{1/2}^{n-2}$, we see that

$$\deg \frac{u_\varepsilon}{|u_\varepsilon|}(\cdot, z) = 1 \quad \text{from } \partial B_{1/2}^2 \text{ to } S^1$$

eventually. As in [42, Lemma 6.11], we deduce that $\Gamma = \pm[\![\{0\} \times B_1^{n-2}]\!]$. We conclude that, after possibly replacing (u, ∇) with the conjugate pair, we have

$$\begin{aligned}
0 &< \lim_{\varepsilon \to 0} \mathbf{E}(u_\varepsilon, \nabla_\varepsilon, B_{1/2}^n, \mathbb{R}^{n-2}) \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{B_1^n} [e_\varepsilon(u_\varepsilon, \nabla_\varepsilon) - J(u_\varepsilon, \nabla_\varepsilon) \wedge e_3^* \wedge \cdots \wedge e_n^*] \\
&= \mu_V(B_{1/2}^n) - \langle \Gamma, \mathbf{1}_{B_{1/2}^n} e_3^* \wedge \cdots \wedge e_n^* \rangle \\
&= 0,
\end{aligned}$$

which is the desired contradiction.

Entire case. For the case of an entire solution (u, ∇), we perform a rescaling: writing $\nabla = d - i\tilde{\alpha}$, let

$$u_\varepsilon(x) := u(\varepsilon^{-1}x), \quad \nabla_\varepsilon := d - i\alpha_\varepsilon \text{ with } \alpha_\varepsilon(x) := \varepsilon^{-1}\alpha(\varepsilon^{-1}x).$$

Again, by applying Theorem 4.1, up to extracting a subsequence we have $e_\varepsilon(u_\varepsilon, \nabla_\varepsilon)\, dx \overset{*}{\rightharpoonup} d\mu_V$, for a stationary integral $(n-2)$-varifold V, and the Jacobians $J(u_\varepsilon, \nabla_\varepsilon) \rightharpoonup \Gamma$ for an integral $(n-2)$-cycle Γ with the pointwise bound $|\Gamma| \leq \mu_V$. Using the monotonicity formula for E_ε, we see that

$$\mu_V(B_R^n) = \lim_{\varepsilon \to 0} \frac{\varepsilon^{n-2}}{2\pi} \int_{B_{R/\varepsilon}^n} e_1(u, \nabla)$$

is a constant multiple of R^{n-2}, and hence V is a cone around the origin with $\mu_V(B_1^n) \leq 1 + \frac{\tau_0}{2\pi}$. Then, by Allard's regularity theorem [2], we see that for $\tau_0(n)$ small enough, after a suitable rotation, V is the varifold associated to $\{0\} \times \mathbb{R}^{n-2}$. In particular, this shows the conclusion on the energy limit.

As before, we also have $\Gamma = \pm[\![\{0\} \times \mathbb{R}^{n-2}]\!]$, concluding that for $R := \varepsilon_k^{-1}$ (where $\varepsilon_k \to 0$ is our subsequence) the statement holds for the plane $S(R) := \{0\} \times \mathbb{R}^{n-2}$, either for (u, ∇) or the conjugate pair $(\overline{u}, \overline{\nabla} = d + i\alpha)$ (depending on R). Since the initial sequence $\varepsilon_k \to 0$ was arbitrary, we deduce that

$$\min_{S\in \mathrm{Gr}(n,n-2)} \min\{\mathbf{E}(u,\nabla,B^n_R,S),\mathbf{E}(\overline{u},\overline{\nabla},B^n_R,S)\}\to 0 \tag{5.10}$$

as $R\to\infty$. Finally, letting $S(R)$ realize the minimum over $S\in \mathrm{Gr}(n,n-2)$, since $\mathbf{E}_1\le \mathbf{E}$ does not distinguish between (u,∇) and $(\overline{u},\overline{\nabla})$ we have

$$\mathbf{E}_1(u,\nabla,B^n_R,S(R))\to 0.$$

As a consequence, we must have

$$\sup_{R'\in[R,2R]} \|P_{S(R')}-P_{S(R)}\|\to 0, \tag{5.11}$$

since otherwise we would find sequences $S(R_k)\to\hat{S}$ and $S(R_k')\to\hat{S}'\neq\hat{S}$ (with $R_k\le R_k'\le 2R_k$) for which

$$\mathbf{E}_1(u,\nabla,B^n_R,\hat{S})+\mathbf{E}_1(u,\nabla,B^n_R,\hat{S}')\to 0 \quad \text{as } R=R_k\to\infty,$$

thanks to the assumption $\int_{B^n_R} e_1(u,\nabla)=O(R^{n-2})$. If $\hat{S}\cup\hat{S}'$ spans $\mathbb{R}^n$, this immediately gives $\int_{B^n_R} e_1(u,\nabla)=o(R^{n-2})$, contradicting the assumption $u(0)=0$ and the clearing-out lemma. Otherwise, their span is $(n-1)$-dimensional; letting e_1 be a unit vector orthogonal to it and completing to an orthonormal basis $\{e_1,\dots,e_n\}$ such that $e_2\perp\hat{S}$, we deduce that

$$\int_{B^n_R}\left[\sum_{j=2}^n |\nabla_{e_j}u|^2+|\omega|^2\right]=o(R^{n-2}).$$

Because of (5.10), we also have

$$\min\{\mathbf{E}_2(u,\nabla,B^n_R,\hat{S}),\mathbf{E}_2(\overline{u},\overline{\nabla},B^n_R,\hat{S})\}\to 0,$$

giving

$$\int_{B^n_R}\left[||\nabla_{e_1}u|-|\nabla_{e_2}u||^2+\left||\omega(e_1,e_2)|-\frac{1-|u|^2}{2}\right|^2\right]\to 0,$$

giving again the contradiction $\int_{B^n_R} e_1(u,\nabla)=o(R^{n-2})$.

Having established (5.11), the claim follows by a straightforward continuity argument: for R large enough we cannot have that

$$\mathbf{E}(u,\nabla,B^n_R,S(R)),\quad \mathbf{E}(\overline{u},\overline{\nabla},B^n_{R'},S(R'))$$

are both small, for some $R'\in[R,2R]$, since this would imply that

$$\mathbf{E}_2(u,\nabla,B^n_R,S(R))+\mathbf{E}_2(\overline{u},\overline{\nabla},B^n_R,S(R))$$

is also small, which would give again small normalized energy on B^n_R; the same holds interchanging the roles of R and R', completing the proof. □

We also record the following consequence of the Hausdorff convergence of the vorticity set $Z=\{|u|\le\frac{3}{4}\}$.

Lemma 5.4 (Soft height bound)**.** *For any $\sigma > 0$ there exist $\tau_0(n,\sigma) > 0$ and $\varepsilon_0(n,\sigma) > 0$ with the following property. Let (u,∇) be a critical point for E_ε on B_1^n, with $\varepsilon \le \varepsilon_0$ and $u(0) = 0$, as well as the energy bound*

$$\frac{1}{|B_1^{n-2}|}\int_{B_1^n} e_\varepsilon(u,\nabla) \le 2\pi + \tau_0$$

and (4.1)–(4.2). Then, after a suitable rotation, the zero set is contained in a small neighborhood of $\mathbb{R}^{n-2}$; more precisely,

$$\{|u_\varepsilon| \le 3/4\} \cap B_{1-\sigma}^n \subset B_\sigma^2 \times B_1^{n-2}.$$

Proof. Following the same strategy as in the proof of Proposition 5.3, the statement follows from the Hausdorff convergence of the vorticity set in Theorem 4.1. □

Remark 5.5. We will often use the following observation: if the excess $\mathbf{E}_1$ is suitably small on B_1^n, then the same conclusion holds without any rotation. The same holds under other assumptions forcing the vorticity set to concentrate on the plane $\mathbb{R}^{n-2}$ in the limit $\varepsilon \to 0$, such as energy close to $|B_1^{n-2}| \cdot 2\pi$ on the cylinder $B_1^2 \times B_1^{n-2}$, for a critical pair defined there (with $u(0) = 0$).

In the following lemma we essentially show that if $\mathbf{E}_1$ is small in a ball of radius larger than ε, then $\mathbf{E}$ is small as well.

Lemma 5.6 ($\mathbf{E}_1$ vanishing implies $\mathbf{E}$ vanishing)**.** *For any $\delta > 0$ there exist $\tau_0(n,\delta) > 0$ and $\varepsilon_0(n,\delta) > 0$ small enough with the following property. Let (u,∇) be a critical pair for E_ε on the unit ball B_1^n, with $u(0) = 0$,*

$$E_\varepsilon(u,\nabla) \le 2\pi + \tau_0,$$

and (4.1)–(4.2), as well as $\varepsilon \le \varepsilon_0$. Let $x \in B_{1-\delta}^n$ be a point such that

$$\sup_{\varepsilon \le s \le 1-|x|} \mathbf{E}_1(u,\nabla,B_s^n(x),\mathbb{R}^{n-2}) \le \tau_0.$$

Then, up to conjugating the pair,

$$\sup_{\varepsilon \le s \le 1-|x|} \mathbf{E}(u,\nabla,B_{s/2}^n(x),\mathbb{R}^{n-2}) \le \delta.$$

Proof. The proof of this lemma is basically the equivalence of the (second-order) Euler–Lagrange equations and the (first-order) vortex equations in two dimensions.

By contradiction, assume we have a sequence (u_k,∇_k) of critical points for E_ε, with $\varepsilon = \varepsilon_k \to 0$, and a sequence of points $x_k \in B_{1-\delta}^n$ and radii $s_k \in [\varepsilon_k, 1-|x_k|]$ such that

$$\mathbf{E}_1(u_k,\nabla_k,B_{s_k}^n(x_k),\mathbb{R}^{n-2}) \to 0, \quad \liminf_{k\to\infty} \mathbf{E}(u_k,\nabla_k,B_{s_k/2}^n(x_k),\mathbb{R}^{n-2}) \ge \delta.$$

We now distinguish a few cases depending on the behavior of the limit ε_k/s_k, which we can assume to exists and to belong to $[0,1]$, and on the distance of x_k from the vorticity set $Z_k = \{x \in B_{s_k}^n(x_k) : |u_k| \le 3/4\}$.

Case 1: $\varepsilon_k/s_k \to 0$ *and* $\operatorname{dist}(x_k, Z_k)/s_k \to 0.$ Since the energy concentration varifold is a plane with multiplicity 1 (as in the previous proof), recalling that $1 - |x_k| \geq \delta$ and x_k has vanishing distance from the vorticity set, we immediately see that

$$\frac{1}{|B^{n-2}_{1-|x_k|}|}\int_{B_{1-|x_k|}(x_k)} e_{\varepsilon_k}(u_k, \nabla_k) \to 2\pi.$$

Defining the map $\phi_k(x) := x_k + s_k x$, we consider the pullback pair

$$(\tilde{u}_k, \tilde{\nabla}_k) := \phi_k^*(u_k, \nabla_k),$$

which is critical for $E_{\tilde{\varepsilon}_k}$, where $\tilde{\varepsilon}_k := \varepsilon_k/s_k$. Moreover, we have

$$\limsup_{k\to\infty} \frac{1}{|B_1^{n-2}|}\int_{B_1^n} e_{\tilde{\varepsilon}_k}(\tilde{u}_k, \tilde{\nabla}_k) \leq 2\pi$$

by monotonicity of the energy.

Since $\tilde{\varepsilon}_k \to 0$ and 0 has vanishing distance from $\{|\tilde{u}_k| \leq \frac{3}{4}\} \cap B_1^n$, as in the previous proof, the energy concentration varifold V is a plane S passing through the origin, with multiplicity 1, while the limiting cycle $\Gamma = \pm[\![S]\!]$. By possibly replacing $(\tilde{u}, \tilde{\nabla})$ with their conjugate, we can assume that $\Gamma = [\![S]\!]$. Also, the stress-energy tensors

$$T_{\tilde{\varepsilon}_k}(\tilde{u}_k, \tilde{\nabla}_k),$$

viewed as matrix-valued measures, converge (up to subsequences) to a limit T such that $dT(x) = P_{T_xV}\, d\mu_V(x)$, where P_{T_xV} is the orthogonal projection onto the tangent space T_xV (cf. [42, Section 6.1]). Hence, the fact that $\mathbf{E}_1(\tilde{u}_k, \tilde{\nabla}_k, B_1^n, \mathbb{R}^{n-2}) \to 0$ implies $T_xV = \mathbb{R}^{n-2}$ a.e., giving $S = \mathbb{R}^{n-2}$. Since

$$\lim_{k\to\infty}\int_{B^n_{1/2}} [e_{\tilde{\varepsilon}_k}(\tilde{u}_k, \tilde{\nabla}_k) - J(\tilde{u}_k, \tilde{\nabla}_k) \wedge e_S^*] = 2\pi[\mu_V(B^n_{1/2}) - \langle \Gamma, \mathbf{1}_{B^n_{1/2}} e_S^*\rangle] = 0$$

, we get the desired contradiction in this case.

Case 2: $\varepsilon_k/s_k \to 0$ *and* $\operatorname{dist}(x_k, Z_k)/s_k \to 2d > 0.$ By applying the same scaling as in the previous step we get that $|\tilde{u}_k|$ converges uniformly to 1 in $B^n_d(0)$, which immediately implies that both excesses converges to 0 in $B^n_{ds_k}(x_k)$ and thus the statement of the theorem with $s/2$ replaced by ds. A covering argument then allows to pass to $s/2$.

Case 3: $\varepsilon_k/s_k \to \overline{\varepsilon} > 0.$ Note that this implies that $s_k \to 0$.

After passing to a local Coulomb gauge, for any $\ell \in \mathbb{N}$ we get local uniform C^ℓ bounds on $B^n_{R_k}$, with $R_k := \delta/s_k$, since by monotonicity we have local uniform bounds on the energy here (see [42, Appendix]). By Arzelà–Ascoli we obtain a subsequential limit $(\tilde{u}_\infty, \tilde{\nabla}_\infty)$ in $C^\infty(\mathbb{R}^n)$. By the definition of $\mathbf{E}_1$ (cf. (5.2)), we see that $(\tilde{\nabla}_\infty)_{\partial_k}\tilde{u}_\infty = 0$ for all $3 \leq k \leq n$ and $\tilde{\omega}_\infty(e_j, e_k) = 0$ for all $(j,k) \neq (1,2)$. As in [42, Proposition 6.7] (after [42, eq. (6.30)]), up to a further change of gauge, the limiting pair depends only on the first two coordinates. By the equivalence of first-order and second-order vortex equations in $\mathbb{R}^2$ [52] (cf. also the end of the proof of [42, Proposition 6.7]), we see that $(\tilde{u}_\infty, \tilde{\nabla}_\infty)$ solves the first-order vortex equations up to conjugation; this yields a contradiction for k large enough. $\square$

Lemma 5.7. *For every $\sigma > 0$ there exist constants $\eta(n,\sigma), C(n,\eta) > 0$ such that if $r \geq C\varepsilon$ and (u,∇) is a critical pair on $B_r(p)$ satisfying (4.1)–(4.2) and $\mathbf{E}_1(u,\nabla,B^n_r(p),\mathbb{R}^{n-2}) \leq \eta$ then*

$$\{|u| \leq 3/4\} \cap B^n_{(1-\sigma)r}(p) \subseteq B^2_{\sigma r}(y) \times \mathbb{R}^{n-2},$$

provided that $|u(p)| \leq \frac{3}{4}$ at $p = (y,z)$ and the normalized energy is at most $2\pi + \eta$.

Moreover, given $\sigma, \Lambda > 0$ there are $\eta(n,\sigma,\Lambda), C(n,\eta,\Lambda) > 0$ such that if

$$\mathbf{E}_1(u,\nabla,B^n_{C\varepsilon}(p),\mathbb{R}^{n-2}) \leq \eta$$

then $G := \{u = 0\} \cap B^n_{\Lambda\varepsilon}(p)$ is a σ-Lipschitz graph, we have the inclusion

$$\{|u| \leq 3/4\} \cap B^n_{\Lambda\varepsilon}(p) \subseteq B_{C(n)\varepsilon}(G),$$

and $\varepsilon|u|$ is comparable with the distance from S in this neighborhood.

Proof. The first part follows by the very same arguments of Lemma 5.6. The second one is again showed by contradiction after scaling by ε, noticing that in the Coulomb gauge the contradicting sequence (u_k, ∇_k) converges smoothly to a solution depending only on the two variables (y_1, y_2). To infer the smooth convergence of the zero set (which is gauge invariant) one notices that, by the explicit form of the Taubes solution, the Jacobian $Ju_k(e_1,e_2)$ is bounded away from zero. Convergence of the zero set then follows from the implicit function theorem. Compare also with the proof of Proposition 6.6. $\square$

In the next lemma we show that the energy on each slice is approximately the excess on the slice plus the degree of u on the boundary.

Lemma 5.8. *Let (u,∇) be an arbitrary smooth pair defined on $\overline{B}^2_1 \times \overline{B}^{n-2}_1$ (not necessarily a critical point) with*

$$e_\varepsilon(u,\nabla)(x) \leq e^{-K/\varepsilon} \quad \text{for all } x \in \partial B^2_1 \times B^{n-2}_1$$

and $|u(x)| \geq \frac{1}{2}$ for all x in the same set. Then we have

$$\left| \deg(u/|u|, \partial B^2_1 \times \{z\}) + \mathbf{E}_z - \frac{1}{2\pi} \int_{B^2_1 \times \{z\}} e_\varepsilon(u,\nabla) \right| \leq 4\varepsilon e^{-K/\varepsilon},$$

for all $z \in B^{n-2}_1$, up to conjugating the pair.

Proof. First of all, by a completion of squares, since

$$J := J(u,\nabla)(e_1,e_2) = 2\langle i\nabla_1 u, \nabla_2 u\rangle + (1-|u|^2)\omega(e_1,e_2),$$

we see that

(5.12)
$$\begin{aligned}
&\frac{1}{2\pi}\int_{B_1^2\times\{z\}} e_\varepsilon(u,\nabla)\\
&=(\mathbf{E}_1)_z+\frac{1}{2\pi}\int_{B_1^2\times\{z\}}\left[|\nabla_1 u|^2+|\nabla_2 u|^2+\varepsilon^2\omega(e_1,e_2)^2+\frac{(1-|u|^2)^2}{4\varepsilon^2}\right]\\
&=(\mathbf{E}_1)_z+\frac{1}{2\pi}\int_{B_1^2\times x}\left[|i\nabla_1 u-\nabla_2 u|^2+\left|\varepsilon\omega(e_1,e_2)-\frac{1-|u|^2}{2\varepsilon}\right|^2+J\right]\\
&=\mathbf{E}_z+\frac{1}{2\pi}\int_{B_1^2\times\{z\}} J.
\end{aligned}$$

We then define the modulus $r : B_1^2 \to [0,\infty)$ and the phase $\theta : B_1^2 \setminus \{r = 0\} \to S^1$ by

$$r(y) := |u(y,z)|, \quad \theta(y) := \frac{u}{|u|}(y,z).$$

Writing $\nabla = d - i\alpha$, we also have

$$r^2(d\theta-\alpha)(y) = \langle \nabla u, iu\rangle(y,z)$$

(note that θ and α are not gauge-invariant). Recalling that $J(u,\nabla) = d\alpha + d\langle\nabla u, iu\rangle$, we compute

$$\int_{B_1^2\times\{z\}} J(u,\nabla) = \int_{\partial B_1^2\times\{z\}} [(1-r^2)\alpha(\tau) + r^2\partial_\tau\theta],$$

where τ is the tangent vector to ∂B_1^2. Hence, we have

$$\int_{B_1^2\times\{z\}} J(u,\nabla) = 2\pi\deg(u/|u|,\partial B_1^2\times\{z\}) + \int_{\partial B_1^2\times\{z\}}(1-r^2)[\alpha(\tau)-\partial_\tau\theta],$$

and the last integrand is bounded by

$$(|u|^{-2}-1)|\langle\nabla u, iu\rangle| \le 4(1-|u|^2)|\nabla u| \le 4\varepsilon e_\varepsilon(u,\nabla)$$

in absolute value. Combining these bounds, the claim follows. □

6. Slicing the current and Lipschitz approximation

In this section, inspired by [5, 37, 19], we slice the currents Γ_ε dual to the Jacobians $J(u,\nabla)$. We get metric-space-valued functions of bounded variation (MBV) in the sense of Ambrosio [3], with values in 0-currents in $\mathbb{R}^2$. Then, by placing a threshold on the maximal function of $\mathbf{E}_1$, we construct a Lipschitz approximation of the barycenter of each slice with a uniform $W^{1,2}$ bound.

6.1. **Slicing identities and BV estimates.** We start by defining vertical slices.

Definition 6.1. We define the *vertical slices* of the current Γ_ε, $(\Gamma_\varepsilon)_z = \langle \Gamma_\varepsilon, P, z\rangle$, by the following identity:

$$\int_{B_1^{n-2}(0)} \langle (\Gamma_\varepsilon)_z, \psi\rangle \phi(z)\, dz = \langle \Gamma_\varepsilon, \psi(y)\phi(z)dz\rangle,$$

for any two functions $\psi \in C_c^\infty(B_1^2)$ and $\phi \in C_c^\infty(B_1^{n-2})$, where $P : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is the projection on the last $n-2$ coordinates.

In the next lemma we derive BV estimates for the slices, given a smooth pair (u, ∇) defined on $B_1^2 \times B_1^{n-2}$.

Lemma 6.2 (BV-type estimate)**.** *Define the function* $\Phi_\psi : B_1^{n-2} \to \mathbb{R}$ *by*

$$\Phi_\psi(z) := \langle (\Gamma_\varepsilon)_z, \psi\rangle. \tag{6.1}$$

Then, assuming $\int_{B_1^2\times B_1^{n-2}} e_\varepsilon(u,\nabla) \le 2\pi\Lambda$, *the total variation of* $\Phi_\psi(x)$ *is bounded by* $\mathbf{E}_1$ *and* $\mathbf{E}$ *as follows:*

$$\frac{1}{2}|D\Phi_\psi|(B_1^{n-2})^2 \le \|d\psi\|^2_{L^\infty}\Lambda \min\{C(n)\mathbf{E}_1, \mathbf{E}\},$$

where $|D\Phi_\psi|$ *denotes the total variation measure, and* $\mathbf{E}$ *and* $\mathbf{E}_1$ *are measured on* $B_1^2 \times B_1^{n-2}$ *(without normalization).*

Proof. The notation and line of argument is inspired from [19, Lemma A.1]. For any $\phi \in C_c^\infty(B_1^{n-2}, \mathbb{R}^{n-2})$ we define the $(n-3)$-form α by

$$\alpha := \sum_{k=3}^{n} (-1)^{k-1}\phi_k(x_3, \dots, x_n)dx_3 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n,$$

so that

$$d\alpha = (\operatorname{div} \phi)(z)dz.$$

Now, writing $x = (y, z)$, we have

$$\begin{aligned}\int_{B_1^{n-2}} \Phi_\psi(z) \operatorname{div}\phi(z)\, dz &= \int_{B_1^{n-2}} \langle (\Gamma_\varepsilon)_z, \psi\rangle (\operatorname{div}\phi)(z)\, dz \\ &= \langle \Gamma_\varepsilon, \psi(y)(\operatorname{div}\phi)(z)dz\rangle \\ &= \langle \Gamma_\varepsilon, d(\psi\alpha)\rangle - \langle \Gamma_\varepsilon, d\psi\wedge\alpha\rangle \\ &= -\langle \Gamma_\varepsilon, d\psi \wedge \alpha\rangle,\end{aligned}$$

where the last equality follows from the fact that $\partial\Gamma_\varepsilon = 0$. Now notice that $d\psi \wedge \alpha$ is a linear combination of $(n-2)$-covectors of the form

$$dx_j \wedge dx_3 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n \quad \text{with } j = 1, 2,\ k = 3, \dots, n.$$

As a consequence,

$$|\langle \Gamma_\varepsilon, d\psi \wedge \alpha\rangle| \le \|d\psi\|_{L^\infty}\|\alpha\|_{L^\infty} \sum_{\substack{j=1,2\\ k=3,\dots,n}} \int_{B_1^2\times B_1^{n-2}} [2|\langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle| + (1-|u|^2)|\omega(e_j,e_k)|],$$

which, by Cauchy–Schwarz, is bounded by

$$\|d\psi\|_{L^\infty}\|\alpha\|_{L^\infty}\cdot C(n)\sqrt{\Lambda}\sqrt{\mathbf{E}_1}.$$

Taking the supremum over the functions ϕ with $\|\phi\|_{L^\infty}\le 1$, we get the BV bound

$$|D\Phi_\psi|(B_1^{n-2}) \le C(n)\|d\psi\|_{L^\infty}\sqrt{\Lambda}\sqrt{\mathbf{E}_1}.$$

We can also estimate in the following way. Set $B := B_1^2\times B_1^{n-2}$ and

$$\vec{e}_{n-2} = e_3\wedge\dots\wedge e_n, \quad e^*_{n-2} := dx_3\wedge\dots\wedge dx_n,$$

and let us write $d\Gamma_\varepsilon = \overrightarrow{\Gamma}_\varepsilon\, d|\Gamma_\varepsilon|$ (viewing Γ_ε as a measure with values in $\Lambda_{n-2}\mathbb{R}^n$). Since $d\psi\wedge\alpha$ does not have any e^*_{n-2}-component, if we write $\vec{\Gamma}_\varepsilon = (\vec{\Gamma}_\varepsilon\cdot\vec{e}_{n-2})\vec{e}_{n-2} + \vec{R}$ (where the dot denotes the scalar product in $\Lambda^{n-2}\mathbb{R}^n$), we get

$$\langle \Gamma_\varepsilon, d\psi\wedge\alpha\rangle = \int_B \vec{R}\cdot(d\psi\wedge\alpha)\, d|\Gamma_\varepsilon|,$$

and moreover

$$\begin{aligned}\int_B |\vec{R}|^2\, d|\Gamma_\varepsilon| &= \int_B (1-(\vec{\Gamma}_\varepsilon\cdot\vec{e}_{n-2})^2)\, d|\Gamma_\varepsilon| \\ &\le 2\int_B (1-\vec{\Gamma}_\varepsilon\cdot\vec{e}_{n-2})\, d|\Gamma_\varepsilon| \\ &= 2e(\Gamma_\varepsilon, B, \vec{e}_{n-2}),\end{aligned}$$

where $e(\Gamma_\varepsilon, B, \vec{e}_{n-2})$ is the *current excess* defined by

$$e(\Gamma_\varepsilon, B, \vec{e}_{n-2}) := \frac{1}{2}\int_B |\overrightarrow{\Gamma}_\varepsilon - \vec{e}_{n-2}|^2\, d|\Gamma_\varepsilon|.$$

Hence,

$$\begin{aligned}|\langle \Gamma_\varepsilon, d\psi\wedge\alpha\rangle| &= \left|\int_B \overrightarrow{R}\cdot(d\psi\wedge\alpha)\, d|\Gamma_\varepsilon|\right| \\ &\le |d\psi\wedge\alpha|\int_B |\vec{R}|\, d|\Gamma_\varepsilon| \\ &\le \|d\psi\|_{L^\infty}\|\alpha\|_{L^\infty}\sqrt{2e(\Gamma_\varepsilon, B, \vec{e}_{n-2})}\sqrt{|\Gamma_\varepsilon|(B)}\,.\end{aligned}$$

Again, taking the supremum over the functions ϕ with $\|\phi\|_{L^\infty}\le 1$, we get

$$|D\Phi_\psi|(B_1^{n-2}) \le \|d\psi\|_{L^\infty}\sqrt{2e(\Gamma_\varepsilon, B, \overrightarrow{e}_{n-2})}\sqrt{|\Gamma_\varepsilon|(B)}.$$

From the pointwise bound of the Jacobian $|\Gamma_\varepsilon| \le \frac{1}{2\pi} e_\varepsilon(u,\nabla)$ we see that

$$e(\Gamma_\varepsilon, B, \overrightarrow{e}_{n-2}) = |\Gamma_\varepsilon|(B) - \langle \Gamma_\varepsilon, \mathbf{1}_B \overrightarrow{e}_{n-2}\rangle \le \mathbf{E}.$$

The previous bounds, together with $|\Gamma_\varepsilon|(B) \le \Lambda$, give the conclusion. □

Remark 6.3. The Jerrard–Soner-type computations in Lemma 6.2 are valid for any current without boundary (formally dual to a closed form). In the case of the Yang–Mills–Higgs Jacobian, we record the following identity for convenience (it will be used in Proposition 6.4 and Proposition 7.2):

$$\begin{aligned}\langle d\Phi_\psi, \phi\rangle = \frac{1}{2\pi}\int_{B_1^2\times B_1^{n-2}} \sum_{j=1,2}\sum_{k=3}^{n}(-1)^j[\langle 2i\nabla_{e_j}u, \nabla_{e_k}u\rangle \\ + (1-|u|^2)\omega(e_j,e_k)]\partial_{e_{3-j}}\psi\,\phi_k,\end{aligned} \tag{6.2}$$

for any $\phi \in C_c^1(B_1^{n-2}, \mathbb{R}^2)$.

6.2. **Lipschitz approximation of the barycenter.** Parallel to the regularity theory of minimal currents, we define a Lipschitz approximation of the barycenter of the slices of Γ_ε (see for instance [19, Lemma A.2]). First we fix some notation which will be used frequently:

- we use $\mathbf{E}_1$ as shorthand for $\mathbf{E}_1(u,\nabla, B_1^2\times B_1^{n-2}, \mathbb{R}^{n-2})$, and similarly for $\mathbf{E}$;
- as already mentioned, for any $z \in B_1^{n-2}$ we denote the excess on the slice $B_1^2 \times \{z\}$ by $(\mathbf{E}_1)_z$, and similarly for $\mathbf{E}_z$;
- we write $M_{\mathbf{E}_1}(z)$ to denote the maximal function of $(\mathbf{E}_1)_z$;
- we fix a cut-off function $\chi \in C_c^\infty(B_{3/4}^2)$ such that $0 \le \chi \le 1$ and $\chi = 1$ on $B_{1/2}^2$.

Proposition 6.4 (Lipschitz approximation)**.** *Given $0 < \eta \le \eta_0(n)$ small enough, there exist $\tau_0(n,\eta) > 0$ and $\varepsilon_0(n,\eta) > 0$ such that the following holds. Let (u,∇) be a critical pair for E_ε, defined on $B_1^2 \times B_1^{n-2}$, satisfying $u(0) = 0$, (4.1)–(4.2), and the energy bound*

$$\frac{1}{|B_1^{n-2}|}\int_{B_1^2\times B_1^{n-2}} e_\varepsilon(u,\nabla) \le 2\pi + \tau_0.$$

Then, up to conjugating (u,∇), for $0 < \eta \le \eta_0(n)$ small enough there exists a Lipschitz approximation $h : B_{3/4}^{n-2} \to \mathbb{R}^2$ with the following properties:

(i) $\mathrm{Lip}(h) \le C\eta$ *and* $\int_{B_{3/4}^{n-2}} |dh|^2 \le C\mathbf{E}_1$*;*

(ii) $h|_{\mathcal{G}^\eta} = \Phi_{\chi(x_1,x_2)}$ *(defined in (6.1)) for a set $\mathcal{G}^\eta \subseteq B_{3/4}^{n-2}$ such that $|B_{3/4}^{n-2} \setminus \mathcal{G}^\eta| \le C\frac{\mathbf{E}_1}{\eta^2}$;*

(iii) $\int_{B_{3/4}^2\times(B_{3/4}^{n-2}\setminus\mathcal{G}^\eta)} e_\varepsilon(u,\nabla) \le C\frac{\mathbf{E}_1}{\eta^2} + e^{-K/\varepsilon}$*;*

(iv) $\int_{\mathcal{G}^\eta} \frac{|dh|^2}{2} \le (1+\delta)\int_{\mathcal{G}^\eta} \mathbf{E}_z\,dz + e^{-K/\varepsilon}$ *with $\delta(n,\eta) > 0$ such that* $\lim_{\eta\to 0}\delta(n,\eta) = 0$.

Here $C = C(n) > 0$ and $K = K(n) > 0$, provided that $\varepsilon \le \varepsilon_0$.

Proof. We define the *good set* to be

$$\mathcal{G}^\eta := \{z \in B^{n-2}_{3/4} \,:\, M_{\mathbf{E}_1}(z) \le \eta^2\}. \tag{6.3}$$

By the weak L^1 bound and Vitali's covering lemma, we can bound the measure of the complement of the good set, namely the *bad set*, by

$$\mathcal{H}^{n-2}(B^{n-2}_{3/4} \setminus \mathcal{G}^\eta) \le C(n)\frac{\mathbf{E}_1}{\eta^2}. \tag{6.4}$$

Bounding energy on the bad set. To check that the third conclusion holds, we introduce another *bad set* $\mathcal{B}^\eta$, defined on the n-dimensional space: it is the set of points $x = (y,z) \in B^2_{3/4} \times B^{n-2}_{3/4}$ such that, for some radius $r \in (0, \frac{1}{50})$, we have $\mathbf{E}_1(u, \nabla, B^n_r(x), \mathbb{R}^{n-2}) > \eta^2$.

By Vitali's covering lemma, we can cover $\mathcal{B}^\eta$ with balls $B_{5r_i}(x_i)$ such that the balls $B_{r_i}(x_i)$ are disjoint and $\mathbf{E}_1(u, \nabla, B_{r_i}(x_i), \mathbb{R}^{n-2}) > \eta^2$. By monotonicity of the energy, the energy on each dilated ball $B_{5r_i}(x_i)$ is at most $C(n) r_i^{n-2}$, giving

$$\begin{aligned}
\int_{\mathcal{B}^\eta} e_\varepsilon(u,\nabla) &\le \sum_i C(n) r_i^{n-2} \\
&\le \sum_i \frac{C(n)}{\eta^2} r_i^{n-2} \mathbf{E}_1(u, \nabla, B_{r_i}(x_i), \mathbb{R}^{n-2}) \\
&\le \frac{C(n)}{\eta^2}\mathbf{E}_1
\end{aligned}$$

(recall that the excess on a ball $B_r(x)$ is normalized by a factor r^{2-n}). Since the measure of $B^{n-2}_{3/4} \setminus \mathcal{G}^\eta$ obeys the same bound, it is enough to show that

$$\int_S e_\varepsilon(u,\nabla) \le C(n)$$

for η small enough, where $S := (B^2_{3/4} \times \{z\}) \setminus \mathcal{B}^\eta$.

We denote by d_Z the distance from the vorticity set $Z = \{|u| \le \frac{3}{4}\}$. As we can see from the proof of Lemma 5.4, its conclusion holds without any rotation in the present situation (as necessarily energy concentrates along $\mathbb{R}^{n-2}$ as $\tau_0, \varepsilon_0 \to 0$). Hence, we can assume that, for any $(y,z) \in S$ on this slice, we have $d_Z(y,z) \ge \frac{1}{200}$ unless $|y| < \frac{1}{100}$.

Given $s \ge \varepsilon$, by Lemma 5.7 we know that if

$$d_Z(y,z), d_Z(y',z) < s,$$

for two points $(y,z), (y',z) \in S$ with $|y|, |y'| < \frac{1}{100}$, then

$$|y - y'| \le Cs,$$

provided that η, ε and ε/s are small enough. With this observation in hand, we can apply Proposition 4.5 (giving $e_\varepsilon(u,\nabla)(y,z) \le Ce^{-K\min\{d_Z(y,z),1/10\}/\varepsilon}$

on S) and the coarea formula to write

$$\int_S e_\varepsilon(u,\nabla) \le \frac{C}{\varepsilon^3}\int_0^2 e^{-Kt/\varepsilon}|\{y\in B^2_{1/100} : d_Z(y,z)<t\}|\,dt + \frac{C}{\varepsilon^3}e^{-K/(200\varepsilon)}.$$

The previous observation says that $\{y\in B^2_{1/100} : d_Z(y,z)<t\}$ is included in a ball of radius $C\max\{t,\varepsilon\}$. We deduce that the last integral is bounded by

$$\frac{C}{\varepsilon^3}\int_0^2 e^{-Kt/\varepsilon}C\max\{t^2,\varepsilon^2\}\,dt \le C,$$

giving the desired bound

$$\int_S e_\varepsilon(u,\nabla)\le C(n).$$

Bounds in terms of $(\mathbf{E}_1)_z$. Now we establish Dirichlet energy bounds for Φ_ψ on the good set, for $\psi\in C^1_c(B^2_{3/4})$. Given $z\in\mathcal{G}^\eta$, we can use Remark 6.3 to bound

$$\begin{aligned}&|d\Phi_\psi|^2(z)\\ &\le C\sum_{j=1,2}\sum_{k=3}^n\left[\int_{B^2_1\times\{z\}}\big(|\langle 2i\nabla_{e_j}u,\nabla_{e_k}u\rangle| + (1-|u|^2)|\omega(e_j,e_k)|\big)\,|\partial_{e_{3-j}}\psi|\right]^2\\ &\le C\|d\psi\|^2_{L^\infty}\left[\int_{B^2_{3/4}\times\{z\}}|\nabla_{e_1}u|^2+|\nabla_{e_2}u|^2+\frac{(1-|u|^2)^2}{\varepsilon^2}\right](\mathbf{E}_1)_z.\end{aligned}$$

Since $z\in\mathcal{G}^\eta$, we have $S=B^2_{3/4}\times\{z\}$ in the previous argument. Thus, the last integral is bounded by $C(n)$. As a consequence,

$$|d\Phi_\psi|^2(z)\le C(n)\|d\psi\|^2_{L^\infty}(\mathbf{E}_1)_z \quad \text{for all } z\in\mathcal{G}^\eta.$$

Bounds in terms of $\mathbf{E}_z$. Also, we can use Lemma 6.2 (cf. [19, Lemma A.2]) to conclude that

$$|d\Phi_{\chi(x_1,x_2)}|^2(z)\le 2\mathbf{E}_z\lim_{r\to 0}\frac{|\Gamma_\varepsilon|(B^2_{1/2}\times B^{n-2}_r(z))}{|B^{n-2}_r|}+Ce^{-K/\varepsilon}.$$

(indeed, this bound follows by applying Lemma 6.2 and its proof with $\psi := \chi(ax_1+bx_2)$ for an arbitrary $(a,b)\in S^1$ and using the fact that this ψ is 1-Lipschitz on $B^2_{1/2}$, outside of which the energy density is exponentially small).

To conclude, we have

$$|\Gamma_\varepsilon|(B^2_{1/2}\times B^{n-2}_r(z))\le |B^{n-2}_r| + \int_{B^{n-2}_r(z)}\mathbf{E}_z + |B^{n-2}_r|e^{-K/\varepsilon}$$

by Lemma 5.8, giving

$$|d\Phi_{\chi(x_1,x_2)}|^2(z)\le 2\mathbf{E}_z(1+\mathbf{E}_z)+Ce^{-K/\varepsilon},$$

where we can actually replace $\mathbf{E}_z$ with the excess on the smaller disk $B^2_{1/2}\times\{z\}$, denoted by $\mathbf{E}_z(B^2_{1/2})$. Now, fixing $L>1$ large, by an obvious variant of Lemma 5.6 we have that $\mathbf{E}_z(B^2_\varepsilon(y))$ is small for all $y\in B^2_{1/2}$ (see also the remark below). Since we can cover the set $\{y\in B^2_{1/2} : d_Z(y,z)\le L\varepsilon\}$ with $C(n,L)$ such disks, we infer that

$$\mathbf{E}_z(B^2_{1/2})\le\delta(n,L,\eta)+\frac{C(n)}{\varepsilon^3}\int_{L\varepsilon}^2 e^{-Kt/\varepsilon}|\{y\in B^2_{1/2} : d_Z(y,z)<t\}|\,dt + \frac{C(n)}{\varepsilon^3}e^{-K/(4\varepsilon)},$$

for some quantity $\delta(n,L,\eta)$ vanishing as $\eta\to 0$. Choosing L suitably large, we deduce that

$$\mathbf{E}_z(B^{n-2}_{1/2})\le\delta(n,\eta)+e^{-K/\varepsilon}$$

for some quantity $\delta(n,\eta)$ vanishing as $\eta\to 0$. The statement follows by extending $h:=\Phi_{\chi(x_1,x_2)}|_{\mathcal{G}_\eta}$ to a function with Lipschitz constant $C(n)\eta$. □

Remark 6.5. As a technical remark, a simple continuity argument as in Proposition 5.3 shows that the possible need of conjugating the pair (u,∇) in Lemma 5.6 happens precisely when the degree of $u/|u|$ along the circle $\partial B^2_{1/2}(0)\times\{0\}$ is -1 instead of 1.

6.3. **Lipschitz approximation of the zero set.** In this part we collect information about the Lipschitz approximation of the zero set. We use compactness arguments similar to [54, Sectoin 5].

Proposition 6.6 (Zero set is Lipschitz on the good set)**.** *For any $\sigma,\delta>0$, there exists $\eta_0(n,\sigma,\delta)$ small enough with the following property. For (u,∇) as in the previous statement, for any $\eta\le\eta_0(\delta,\sigma)$, the set $u^{-1}(0)\cap(B^2_{3/4}\times\mathcal{G}^\eta)$ is included in a δ-Lipschitz graph $h_0 : B^{n-2}_{3/4}\to B^2_\sigma$ with the following estimate:*

$$\int_{B^{n-2}_{3/4}}|h_0-h|^2\le C\sigma^2\frac{\mathbf{E}_1}{\eta^2}+C\varepsilon^2|\log(\mathbf{E}_2)|^2\mathbf{E}_2+Ce^{-K/\varepsilon},$$

for $C=C(n)$ and h defined in Proposition 6.4 (provided that $\varepsilon\le\varepsilon_0(n,\sigma,\delta)$ and the energy is $\le 2\pi+\tau_0(n,\sigma,\delta)$).

Proof. The proof is similar to [54, Lemma 5.3].

Lipschitz approximation at scale ε. This is essentially the second part of Lemma 5.7, but we present a detailed argument here. Notice that, locally at scale ε, critical points enjoy uniform C^k estimates in the Coulomb gauge (and thus C^k bounds for gauge-invariant quantities): see [42, Appendix]. Then around any $x_0=(y_0,z_0)\in B^2_{3/4}\times\mathcal{G}^\eta$ with $u(x_0)=0$ we rescale as follows:

$$\tilde u(x):=u(x_0+\varepsilon x),\quad \tilde\nabla:=\phi^*_{x_0,\varepsilon}(\nabla),$$

where $\phi_{x_0,\varepsilon}$ is the map $x \mapsto x_0 + \varepsilon x$. The resulting pair $(\tilde{u}, \tilde{\nabla})$ satisfies

$$\sup_{r \le 1/(4\varepsilon)} \mathbf{E}_1(\tilde{u}, \tilde{\nabla}, B^2_{1/(4\varepsilon)} \times B^{n-2}_r, \mathbb{R}^{n-2}) \le \eta^2$$

(where the excess is normalized by a factor r^{2-n}). By Arzelà–Ascoli we conclude that, for small enough η, $(\tilde{u}, \tilde{\nabla})$ is C^1-close to a pair (u_0, ∇_0) that satisfies the Yang–Mills–Higgs equations (3.2)–(3.3) and depends only on the variables x_1, x_2 (as in the proof of Lemma 5.6). As noted in the proof of Proposition 5.3, $u_0/|u_0|$ has degree ± 1 on large circles, and $u(\cdot, z_0)|_{B^2_{3/4}}$

By the main result of [51, 52], we deduce that (u_0, ∇_0) is the standard entire solution of degree ± 1, centered to vanish just at the origin. For this solution, we have

$$|Ju_0|(0) > 0, \tag{6.5}$$

where Ju_0 is the Jacobian of u_0 in the local Coulomb gauge in B^n_1. It then follows that, for small enough $\eta > 0$, we have $|J\tilde{u}(e_1, e_2)| \ge c > 0$. Then, by an application of the implicit function theorem and the fact that $\{\tilde{u} = 0\}$ is a gauge-invariant set, we see that $\{\tilde{u} = 0\}$ is locally a Lipschitz graph with a (qualitatively) small Lipschitz constant. The fact that the zero set intersects the slice only at x_0 follows from Lemma 5.7, which says that there is no zero outside a $C\varepsilon$-neighborhood of x_0, while in this neighborhood uniqueness follows from the fact that it holds for u_0 (see also a similar argument in the proof of [31, Theorem 4.1]). Hence, for small enough η, we can define a function $h_0 : \mathcal{G}^\eta \to \mathbb{R}^2$ such that

$$\{u = 0\} \cap (B^2_{3/4} \times \mathcal{G}^\eta) = \mathrm{graph}(h_0).$$

Lipschitz approximation at larger scales. By the first part of Lemma 5.7, we see that given two points $(y, z), (y', z') \in \{u = 0\} \cap (B^2_{3/4} \times \mathcal{G}^\eta)$ we have

$$|z - z'| \le \delta |y - y'| \quad \text{if } |y - y'| \ge C(n, \delta)\varepsilon,$$

for a constant $\delta = \delta(\eta) > 0$ such that

$$\delta(\eta) \to 0 \quad \text{as } \eta \to 0.$$

Together with the previous control at scales comparable with ε, this tells us that h_0 is indeed Lipschitz, with $\mathrm{Lip}(h_0)$ vanishing as $\eta \to 0$. We apply the classical extension theorem to build a Lipschitz extension of h_0 defined on B^{n-2}_s.

L^2 estimates. Using the soft height bound of Lemma 5.4 (note that no rotation is needed in the present situation, as necessarily the energy concentrates along $\mathbb{R}^{n-2}$), we have

$$|h| + |h_0| \le \sigma$$

for η (and hence ε) small enough. Using the estimates of Lemma A.1 on the good set $\mathcal{G}^\eta$ (see also Remark A.3) and the measure bound for the bad set

$B^{n-2}_{3/4} \setminus \mathcal{G}^\eta$ we see that

$$\int_{B^{n-2}_{3/4}} |h_0 - h|^2 \le \int_{\mathcal{G}^\eta} |h_0 - h|^2 + \int_{B^{n-2}_{3/4} \setminus \mathcal{G}^\eta} |h_0 - h|^2$$
$$\le C\varepsilon^2 |\log \mathbf{E}_2|^2 \mathbf{E}_2 + C\sigma^2 \frac{\mathbf{E}_1}{\eta^2} + Ce^{-K/\varepsilon}.$$

We thus get the desired conclusion. □

Remark 6.7. We remark that the function h_0 is well-behaved under small rotations, since the construction also rotates. However, the Lipschitz approximation of the slice barycenters, a priori, might not behave well under rotations.

7. Harmonic approximation and a Caccioppoli-type estimate

7.1. **Harmonic approximation.** In this section we show that the Lipschitz approximation of Proposition 6.4 nearly satisfies the Laplace equation. We achieve this by relating the stress-energy tensor to the slices of Γ_ε using the self-dual discrepancy excess $\mathbf{E}_2$. Then we use this with uniform $W^{1,2}$ bounds to show that the Lipschitz approximation is well approximated in L^2 by a harmonic function. To begin with, we state a very well-known lemma.

Lemma 7.1. *For any $\nu > 0$ small there exists $\tau(n,\nu) > 0$ with the following property. Let f be a function in $W^{1,2}(B^n_1)$ such that*

$$\int_{B^n_1} |\nabla f|^2 \le 1, \qquad \left| \int_{B^n_1} \langle df, d\phi \rangle \right| \le \tau \|d\phi\|_{L^\infty},$$

for any $\phi \in C^1_c(B^n_1)$. Then there exists a harmonic function $w : B^n_1 \to \mathbb{R}$ such that

$$\int_{B^n_1} |dw|^2 \le 1, \qquad \int_{B^n_1} |w - f|^2 \le \nu.$$

Moreover, if f has zero average, we can choose w so that $w(0) = 0$.

Proof. The claim follows easily from Rellich's compact embedding theorem: see for instance [20, Lemma 6.1]. For the second part, by the mean value property of harmonic functions and $\int f = 0$ one gets that

$$|B^n_1||w(0)| = \left| \int_{B_1} w - \int_{B^n_1} f \right| \le C(n)\|w - f\|_{L^2} \le \sqrt{\nu}.$$

The function $w - w(0)$ satisfies the conclusion of the lemma. □

Proposition 7.2 (Harmonic approximation)**.** *Let (u, ∇) be a critical point of E_ε as in the previous section and let $h : B^{n-2}_{3/4}$ be the Lipschitz approximation* *built in Proposition 6.4 for η. Then there exist constants $C(n), K(n) > 0$ such that, for any test function $\phi \in C_c^\infty(B_{3/4}^{n-2}, \mathbb{R}^2)$, we have*

$$\left| \int_{B_{3/4}^{n-2}} \langle dh, d\phi \rangle \right| \leq C(\eta^{-1}\mathbf{E}_1 + \sqrt{\mathbf{E}\mathbf{E}_1} + e^{-K/\varepsilon}) \|d\phi\|_{L^\infty}.$$

Moreover, given any $\nu > 0$, if $e^{-K/\varepsilon} \leq \mathbf{E}_1$ and $\mathbf{E}$ is small enough (depending on n, η, ν), there exists a harmonic function $w : B_{3/4}^{n-2} \to \mathbb{R}^2$ with $w(0) = 0$ such that

$$\int_{B_{3/4}^{n-2}} |dw|^2 \leq C, \quad \int_{B_{3/4}^{n-2}} |(\mathbf{E}_1)^{-1/2}(h - c) - w|^2 \leq \nu,$$

where c is the average of h.

Proof. First, we define the vector field $X := \phi(x_3, \dots, x_n)e_1$ for any compactly supported test function $\phi \in C_c^\infty(B_{3/4}^{n-2})$, and we test (4.10) with $\psi(x_1, x_2)X$, where ψ is a smooth cut-off function such that $\psi = 1$ on $B_{1/2}^2$ and $\psi = 0$ outside of $B_{3/4}^2$. We obtain

$$\left| \int_{B_{1/2}^2 \times B_{3/4}^{n-2}} \langle T_\varepsilon(u, \nabla), DX \rangle \right| \leq C e^{-K/\varepsilon} \|d\phi\|_{L^\infty},$$

thanks to the fact that $d\psi$ is supported in the annulus $B_{3/4}^2 \setminus B_{1/2}^2$ and the exponential decay away from the vorticity set Z, which intersects $B_{3/4}^2 \times B_{3/4}^{n-2}$ only inside $B_{1/4}^2 \times B_{3/4}^{n-2}$. Then, since DX is traceless, we compute

$$\begin{aligned} &\int_{B_{1/2}^2 \times B_{3/4}^{n-2}} \langle T_\varepsilon(u, \nabla), DX \rangle \\ &= -2 \int_{B_{1/2}^2 \times B_{3/4}^{n-2}} \sum_{k=3}^n \left[\langle \nabla_{e_1} u, \nabla_{e_k} u \rangle + \varepsilon^2 \sum_{j=1}^n \omega(e_1, e_j)\omega(e_k, e_j) \right] \partial_{e_k} \phi. \end{aligned}$$

Except for $j = 2$, the integral of the terms involving the curvature ω is bounded by $C(n)\mathbf{E}_1$, giving

$$\begin{aligned} &\left| \int_{B_{1/2}^2 \times B_{3/4}^{n-2}} \sum_{k=3}^n [\langle \nabla_{e_1} u, \nabla_{e_k} u \rangle + \varepsilon^2 \omega(e_1, e_2)\omega(e_k, e_2)] \partial_{e_k} \phi \right| \\ &\leq C(\mathbf{E}_1 + e^{-K/\varepsilon}) \|d\phi\|_{L^\infty}. \end{aligned} \tag{7.1}$$

We now want to relate the expression in the left-hand side with the identity for $d\Phi_{\chi x_1}$ obtained in Remark 6.3, which in particular gives

$$\begin{aligned}&\left|\int_{B^{n-2}_{3/4}}\langle d\Phi_{\chi x_1}, d\phi\rangle\right|\\&\le C\left|\int_{B^2_{1/2}\times B^{n-2}_{3/4}}\sum_{k=3}^n[\langle 2i\nabla_{e_2}u,\nabla_{e_k}u\rangle+(1-|u|^2)\omega(e_2,e_k)]\partial_{e_k}\phi\right|\\&\quad+Ce^{-K/\varepsilon}\|d\phi\|_{L^\infty},\end{aligned}$$

or equivalently

$$\begin{aligned}&\left|\int_{B^{n-2}_{3/4}}\langle d\Phi_{\chi x_1}, d\phi\rangle\right|\\&\le C\left|\int_{B^2_{1/2}\times B^{n-2}_{3/4}}\sum_{k=3}^n\left[-\langle i\nabla_{e_2}u,\nabla_{e_k}u\rangle+\frac{1-|u|^2}{2}\omega(e_k,e_2)\right]\partial_{e_k}\phi\right|\\&\quad+Ce^{-K/\varepsilon}\|d\phi\|_{L^\infty}.\end{aligned}\tag{7.2}$$

We observe that the two integrals in (7.1) and (7.2) differ by the integral of

$$\sum_{k=3}^n\left[\langle\nabla_{e_1}u+i\nabla_{e_2}u,\nabla_{e_k}u\rangle+\left(\omega(e_1,e_2)-\frac{1-|u|^2}{2}\right)\omega(e_k,e_2)\right]\partial_{e_k}\phi.$$

Hence, recalling the definition of $\mathbf{E}_2$ and using Cauchy–Schwarz, we conclude that

$$\left|\int_{B^{n-2}_{3/4}}\langle d\Phi_{\chi x_1}, d\phi\rangle\right|\le C(n)(\mathbf{E}_1+\sqrt{\mathbf{E}_1\mathbf{E}}+e^{-K/\varepsilon})\|d\phi\|_{L^\infty}.$$

Repeating the same for $\Phi_{\chi x_2}$, we arrive at the same conclusion for $\Phi_{\chi(x_1,x_2)}$, integrated against any $\phi\in C^\infty_c(B^{n-2}_{3/4},\mathbb{R}^2)$. To conclude we note that thanks to items (i) and (ii) of Proposition 6.4,

$$\int_{B^{n-2}_{3/4}\setminus\mathcal{G}^\eta}|dh|\le C\eta|B^{n-2}_{3/4}\setminus\mathcal{G}^\eta|\le C\frac{\mathbf{E}_1}{\eta}$$

and, in view of Remark 6.3, Cauchy–Schwarz, item (iii) of Proposition 6.4 and the assumption $e^{-K/\varepsilon}\le\mathbf{E}_1$,

$$\int_{B^{n-2}_{3/4}\setminus\mathcal{G}^\eta}|d\Phi_{\chi(x_1,x_2)}|\le C\sqrt{\mathbf{E}_1}\left(\int_{B^2_{3/4}\times B^{n-2}_{3/4}\setminus\mathcal{G}^\eta}e_\varepsilon(u,\nabla)\right)^{1/2}\le C\frac{\mathbf{E}_1}{\eta}.$$

The second part follows from Lemma 7.1, noting that the normalized function $\tilde h:=(\mathbf{E}_1)^{-1/2}h$ has Dirichlet energy bounded by $C(n)$ by item (i) of Proposition 6.4. □

7.2. **Caccioppoli-type estimates.** The starting point in the regularity theory of elliptic partial differential equations is the Caccioppoli–Leray bound, obtained by testing the equation with $\phi^2 u$, where ϕ is a cut-off function and u is the solution. We aim to do something similar in spirit. Here the *function* that we deal with is the barycenter of the energy measure at any slice. This suggests that testing the stress-energy tensor with $\phi^2 x_1 e_1$ and $\phi^2 x_2 e_2$ is an appropriate choice.

Proposition 7.3. *Let (u, ∇) be a critical point of E_ε as in the previous section (with the bounds (4.1) and (4.2)). For any $\sigma > 0$ there exist $\varepsilon_0(n, \sigma)$ and $\tau_0(n, \sigma)$ small enough such that the following Caccioppoli-type estimate holds:*

$$\begin{aligned}&\int_{B^{n-2}_{3/4}} \phi^2(z)(\mathbf{E}_1)_z\,dz\\&\le C\int_{B^2_{1/2}\times B^{n-2}_{3/4}} (x_1^2+x_2^2)e_\varepsilon(u,\nabla)\Delta\phi^2 + C(\sigma^2\mathbf{E}_1 + e^{-K/\varepsilon})\|D^2\phi\|_\infty,\end{aligned}$$

for any test function $\phi \in C^\infty_c(B^{n-2}_{3/4})$, where $C = C(n)$ and $K = K(n, \sigma)$.

Proof. First, we define the vector fields

$$X := \sum_{k=3}^n \partial_k\phi^2(x_3,\dots,x_n)\frac{x_1^2+x_2^2}{2}e_k, \quad Y := \phi^2(x_3,\dots,x_n)(x_1e_1+x_2e_2)$$

and we calculate their derivatives:

$$\begin{aligned}DX &= \frac{1}{2}\sum_{3\le j,k\le n}\partial^2_{e_j,e_k}\phi^2(x_1^2+x_2^2)e_j\otimes e_k^* + \sum_{k=3}^n \partial_{e_k}\phi^2(x_1e_k\otimes e_1^* + x_2e_k\otimes e_2^*)\\DY &= \phi^2(e_1\otimes e_1^* + e_2\otimes e_2^*) + \sum_{k=3}^n\partial_{e_k}\phi^2(x_1e_1\otimes e_k^* + x_2e_2\otimes e_k^*).\end{aligned}$$

Then we test (4.9) with χX and χY, where $\chi = \chi(x_1, x_2)$ is a smooth cut-off function such that $\chi = 1$ on $B^2_{1/2}$ and $\chi = 0$ on $B^2_1 \setminus B^2_{3/4}$. We note that the terms containing $d\chi$ are supported in $(B^2_{3/4}\setminus B^2_{1/2})\times B^{n-2}_{3/4}$, where $|T_\varepsilon(u,\nabla)| \le C(n)e_\varepsilon(u,\nabla)$ is very small by the exponential decay. Hence,

$$\left|\int_{B^2_{1/2}\times B^{n-2}_{3/4}}\langle T_\varepsilon(u,\nabla), DY\rangle - \langle T_\varepsilon(u,\nabla), DX\rangle\right| \le C\|D^2\phi\|_{L^\infty}e^{-K/\varepsilon}.$$

Using the previous expansion of DX and DY, together with the symmetry of $T_\varepsilon(u,\nabla)$, we see that the above integrand equals

$$\frac{1}{2}\sum_{3\le j,k\le n} T_\varepsilon(u,\nabla)(e_j,e_k)\partial^2_{e_j,e_k}\phi^2(x_1^2+x_2^2) - \sum_{j=1,2} T_\varepsilon(u,\nabla)(e_j,e_j)\phi^2$$
$$= \frac{x_1^2+x_2^2}{2}\left[e_\varepsilon(u,\nabla)\Delta\phi^2 - 2\sum_{3\le j,k\le n}(\nabla u^*\nabla u+\varepsilon^2\omega^*\omega)(e_j,e_k)\partial^2_{e_j,e_k}\phi^2\right]$$
$$-2\left[e_\varepsilon(u,\nabla) - |\nabla_{e_1}u|^2 - |\nabla_{e_2}u|^2 - \sum_{j=1,2}\sum_{k=1}^n \varepsilon^2\omega(e_j,e_k)^2\right]\phi^2.$$

By the Modica-type inequality (4.2), the last expression multiplying $-2\phi^2$ is bounded below by

$$\sum_{k=3}^n |\nabla_{e_k}u|^2 + \frac{(1-|u|^2)^2}{4\varepsilon^2} - \varepsilon^2\omega(e_1,e_2)^2 \ge \sum_{k=3}^n |\nabla_{e_k}u|^2 + \varepsilon^2 \sum_{(j,k)\ne(1,2)} \omega(e_j,e_k)^2$$

(where the last sum is over all pairs $(j,k)\ne(1,2)$ with $j<k$), which is the integrand in the definition of $\mathbf{E}_1$. Hence, combining the previous bounds, we arrive at

$$\begin{aligned}
&\int_{B^2_{1/2}\times B^{n-2}_{3/4}} \phi^2(z)(\mathbf{E}_1)_z\,dz \\
&\le \int_{B^2_{1/2}\times B^{n-2}_{3/4}} \frac{x_1^2+x_2^2}{2} e_\varepsilon(u,\nabla)\Delta\phi^2 \\
&\quad + C\int_{B^2_{1/2}\times B^{n-2}_{3/4}} \frac{x_1^2+x_2^2}{2}\sum_{3\le j,k\le n}(\nabla u^*\nabla u+\varepsilon^2\omega^*\omega)(e_j,e_k)\partial^2_{e_j,e_k}\phi^2 \\
&\quad + C\|D^2\phi\|_{L^\infty}e^{-K/\varepsilon}.
\end{aligned}$$

Now, by the soft height bound, we can assume that the vorticity set Z intersects $B^2_{1/2}\times B^{n-2}_{3/4}$ in a small cylinder $B^2_\sigma\times B^{n-2}_{3/4}$; the conclusion follows by exponential decay, up to replacing K with another constant $K(n,\sigma)$. □

Remark 7.4. In the statement of Proposition 7.3 we can replace the first term of the right-hand side as follows:

$$\int_{B^{n-2}_{3/4}} \phi^2(z)(\mathbf{E}_1)_z\,dz \le C\int_{B^2_{1/2}\times B^{n-2}_{3/4}} [(x_1-c_1)^2+(x_2-c_2)^2]e_\varepsilon(u,\nabla)\Delta\phi^2 + C(\sigma^2\mathbf{E}_1 + e^{-K/\varepsilon})\|D^2\phi\|_\infty,$$

provided that $|c|\le C\sigma$ for $C=C(n)$. The proof is essentially the same.

8. Proof of decay of the tilt-excess

In this section we prove Theorem 1.4: roughly speaking, we prove that $\mathbf{E}_1$, the first part of the excess, decays up to scales where it becomes comparable with ε^2. We will deduce this from the excess decay property of harmonic functions, stated in the next elementary lemma.

Lemma 8.1. *Given a harmonic function $w : B_1^n(0) \to \mathbb{R}$, we have the decay estimate*

$$\sup_{x \in B_\rho^n(0)} |w(x) - w(0) - dw(0)[x]| \le C(n)\rho^2 \|dw\|_{L^2}, \tag{8.1}$$

for $\rho \in (0, \frac{1}{2})$.

Proof. By a Taylor expansion, the left-hand side is bounded by the quantity $\frac{\rho^2}{2}\|D^2 w\|_{L^\infty(B_{1/2}^n)}$, which is bounded by $C(n)\rho^2\|dw\|_{L^2}$ by the mean-value property of harmonic functions. □

8.1. **Proof of the excess decay in the case of small $|dw(0)|$.** Take w to be the harmonic approximation built in Proposition 7.2. First, we prove Theorem 1.4 when the harmonic approximation has $|dw(0)| \le \delta$, for a small $\delta > 0$ to be chosen later. We dilate the ball B_1^n to $B_{\sqrt{2}}^n$ (and replace ε with $\varepsilon/\sqrt{2}$), in such a way that it includes $B_1^2 \times B_1^{n-2}$; we also assume that $S = \mathbb{R}^{n-2}$ in the statement.

Let c be the average of h on the ball $B_{3/4}^{n-2}$. The construction of h shows that

$$|c| \le C\sigma + Ce^{-K/\varepsilon} \le C\sigma$$

for ε small enough (depending on n, σ).

We apply the Caccioppoli-type estimates in Proposition 7.3, with $x_1 - c_1$ and $x_2 - c_2$ in place of x_1 and x_2 (see Remark 7.4). Taking $\phi \in C_c^\infty(B_{2\rho}^{n-2})$ to be a cut-off function with $\phi = 1$ on B_ρ^{n-2} and $|D^2\phi| \le C(n)\rho^{-2}$ we get

$$\begin{aligned}
&\int_{B_{3/4}^{n-2}} \phi^2(z)(\mathbf{E}_1)_z \, dz \\
&\le C \int_{B_{1/2}^2 \times B_{3/4}^{n-2}} e_\varepsilon(u, \nabla)[(x_1 - c_1)^2 + (x_2 - c_2)^2]\Delta\phi^2 \\
&\quad + C\rho^{-2}(\sigma^2 \mathbf{E}_1 + e^{-K/\varepsilon}).
\end{aligned}$$

The contribution of the bad set $B_{3/4}^{n-2} \setminus \mathcal{G}^\eta$ can be bounded using the soft height bound of Lemma 5.4 and energy estimate on the bad set (item (iii) in Proposition 6.4), obtaining

$$\begin{aligned}
&\int_{B_{1/2}^2 \times (B_{3/4}^{n-2} \setminus \mathcal{G}^\eta)} e_\varepsilon(u, \nabla)[(x_1 - c_1)^2 + (x_2 - c_2)^2]\Delta\phi^2 \\
&\le C\rho^{-2}(\sigma^2 + e^{-K/\varepsilon})\frac{\mathbf{E}_1}{\eta^2}.
\end{aligned}$$

On the good set $\mathcal{G}^\eta$, we apply Lemma A.2 to estimate the *second moment* of good slices as follows:

$$\begin{aligned}&\left|\int_{B^2_{1/2}\times\mathcal{G}^\eta} e_\varepsilon(u,\nabla)[(x_1-c_1)^2+(x_2-c_2)^2]\Delta\phi^2-\int_{\mathcal{G}^\eta}\varepsilon^2 v_0\,\Delta\phi^2\right|\\&\le C\rho^{-2}\left[\varepsilon^2|\log\mathbf{E}_2|^2\mathbf{E}_2^{1/2}+\sigma^2\mathbf{E}_1+\int_{\mathcal{G}^\eta}|h-c|^2(\mathbf{E}_2)_z^{1/2}+e^{-K\sigma/\varepsilon}\right]\end{aligned}\tag{8.2}$$

(see also Remark A.3), where h is the Lipschitz approximation obtained in Proposition 6.4. Note that the term containing v_0 disappears once integrated on $B^{n-2}_{2\rho}$, as v_0 is a constant and $\Delta\phi^2$ has zero integral.

Combining the previous bounds, we arrive at

$$\begin{aligned}&\left|\int_{B^{n-2}_{3/4}}\phi^2(z)(\mathbf{E}_1)_z\,dz\right|\\&\le C\rho^{-2}\int_{B^{n-2}_{2\rho}}|h-c|^2\\&\quad+C\rho^{-2}\left[(\sigma^2+\varepsilon^2)\frac{\mathbf{E}_1}{\eta^2}+\left(1+\frac{\mathbf{E}_1}{\eta^2}\right)e^{-K/\varepsilon}+\varepsilon^2|\log\mathbf{E}_2|^2\mathbf{E}_2^{1/2}\right].\end{aligned}$$

Assuming $e^{-K/\varepsilon}\le\mathbf{E}_1$, we now apply Proposition 7.2 and Lemma 8.1. Since $\|(h-c)-\sqrt{\mathbf{E}_1}w\|^2_{L^2}\le\nu\mathbf{E}_1$, we have

$$\int_{B^{n-2}_{2\rho}}|h-c|^2\le 2\nu\mathbf{E}_1+2\mathbf{E}_1\int_{B^{n-2}_{2\rho}}|w|^2\le 2\nu\mathbf{E}_1+C\mathbf{E}_1(\rho^{4+(n-2)}+\delta^2\rho^{2+(n-2)}).$$

Thus, for some $C=C(n)$ and $K=K(n,\sigma)$, we get

$$\begin{aligned}&\rho^{2-n}\int_{B^{n-2}_\rho}\mathbf{E}_1(z)\\&\le C\mathbf{E}_1(\rho^{-n}\nu+\rho^2+\delta^2)\\&\quad+C\rho^{-n}\left[(\sigma^2+\varepsilon^2)\frac{\mathbf{E}_1}{\eta^2}+\left(1+\frac{\mathbf{E}_1}{\eta^2}\right)e^{-K/\varepsilon}+\varepsilon^2|\log\mathbf{E}_2|^2\mathbf{E}_2^{1/2}\right].\end{aligned}$$

We now choose η,ρ and, *subsequently*, δ,σ,ν to be small enough. The claim follows (with the same plane $\overline{S}=\mathbb{R}^{n-2}$) once we assume that $\mathbf{E}_1$ is small enough.

8.2. **Tilting the picture.** In the general case, before using the Caccioppoli-type estimate, we need to tilt the picture slightly to ensure that $|dw|(0)$ is small enough. We assume that $\mathbb{R}^{n-2}$ minimizes $\mathbf{E}_1(u,\nabla,B^n_1,\cdot)$.

Consider a rotation $R\in SO(n)$ bringing $\mathbb{R}^{n-2}$ to the graph of the linear map $\sqrt{\mathbf{E}_1}dw(0)$. Since w is harmonic with the bound $\|dw\|_{L^2}\le C$, we have $|\sqrt{\mathbf{E}_1}dw(0)|\le C\sqrt{\mathbf{E}_1}$. Hence, we can find R such that

$$\|R-I\|\le C\mathbf{E}_1^{1/2},\quad \|(P_{\mathbb{R}^{n-2}}\circ R-I)\circ P_{\mathbb{R}^{n-2}}\|\le C\mathbf{E}_1,\tag{8.3}$$

for a dimensional constant $C = C(n)$: indeed, calling S the graph of $\sqrt{\mathbf{E}_1}dw(0)$, using the spectral theorem we can find an orthonormal basis $\{v_3, \dots, v_n\}$ of $\mathbb{R}^{n-2}$ such that the vectors $P_S(v_i)$ form an orthogonal basis of S, so that $\langle P_S(v_i), v_j\rangle = \langle P_S(v_i), P_S(v_j)\rangle = 0$ for $i \neq j$. Thus, $P_{\mathbb{R}^{n-2}} \circ P_S(v_i)$ is parallel to v_i and

$$\frac{P_S(v_i)}{|P_S(v_i)|} = \frac{(\sqrt{\mathbf{E}_1}dw(0)[v_i], v_i)}{\sqrt{1 + \mathbf{E}_1|dw(0)[v_i]|^2}}$$

(under the identification $\mathbb{R}^{n-2} = \{0\} \times \mathbb{R}^{n-2}$).

We extend $\{v_3, \dots, v_n\}$ to an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$. The desired rotation is obtained by sending v_i to $\frac{P_S(v_i)}{|P_S(v_i)|}$ for $i \geq 3$ and v_1, v_2 to suitable unit vectors $v_1 + O(\sqrt{\mathbf{E}_1})$ and $v_2 + O(\sqrt{\mathbf{E}_1})$, obtained for instance via the Gram–Schmidt algorithm on the collection $\{\frac{P_S(v_3)}{|P_S(v_3)|}, \dots, \frac{P_S(v_n)}{|P_S(v_n)|}, v_1, v_2\}$. For $i \geq 3$, since $|P_{S^\perp} v_i| \leq C\sqrt{\mathbf{E}_1}$ we have $|P_S v_i| \geq 1 - C\mathbf{E}_1$, and hence the previous formula gives

$$R(v_i) = R(0, v_i) = (\sqrt{\mathbf{E}_1}dw(0)[v_i], v_i) + O(\mathbf{E}_1) \quad \text{for } i \geq 3. \tag{8.4}$$

Then we define the rotated pair $(\tilde{u}, \tilde{\nabla})$ as follows:

$$\tilde{u} := R^* u, \quad \tilde{\nabla} := R^* \nabla. \tag{8.5}$$

First we prove that the excess changes proportionally after this rotation.

Lemma 8.2 (Tilted excess estimate)**.** *There exists a dimensional constant $C(n)$ such that, for a pair (u, ∇) as in Theorem 1.4 with small enough $\tau_0, \varepsilon_0 > 0$ and a rotation R as above, the tilted excess is bounded by the initial excess; more precisely,*

$$\mathbf{E}_1(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) \leq C\mathbf{E}_1, \quad \mathbf{E}_2(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) \leq C\mathbf{E}. \tag{8.6}$$

Proof. Take an orthonormal basis $e_1, e_2, \dots, e_n$ for $\mathbb{R}^n$ such that $\{e_3, \dots, e_n\}$ is an orthonormal basis for $\mathbb{R}^{n-2}$. Then, recalling the definition of the excess $\mathbf{E}_1$, we have

$$\begin{aligned}\mathbf{E}_1(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) &= \int_{B_1^n} \left[\sum_{k=3}^n |\nabla_{Re_k} u|^2 + \sum_{(j,k)\neq(1,2)} \varepsilon^2 \omega(Re_j, Re_k)^2\right] \\ &\leq C\mathbf{E}_1 + C\|R - I\|^2 E_\varepsilon(u, \nabla) \\ &\leq C\mathbf{E}_1.\end{aligned}$$

The second line above follows from the elementary bounds

$$|\nabla_{Re_k} u - \nabla_{e_k} u| \leq \|R - I\||\nabla u|$$

and

$$|\omega(Re_j, Re_k) - \omega(e_j, e_k)| \leq 2\|R - I\||\omega|.$$

We estimate $\mathbf{E}_2$ in a similar way:

$$\begin{aligned}
&\mathbf{E}_2(\tilde{u}, \tilde{\nabla}, B_1^n, \mathbb{R}^{n-2}) \\
&= \int_{B_1^n} \left[|\nabla_{Re_1} u + i\nabla_{Re_2} u|^2 + \left| \varepsilon\omega(Re_1, Re_2) - \frac{1-|u|^2}{2\varepsilon} \right|^2 \right] \\
&\le C\mathbf{E}_2 + C\|R - I\|^2 E_\varepsilon(u, \nabla) \\
&\le C(\mathbf{E}_1 + \mathbf{E}_2).
\end{aligned}$$

This is indeed the desired conclusion. □

Then we claim that the Lipschitz approximations h and $\tilde{h}$ are approximately a rotation of one another. To do this, we first notice that the Lipschitz approximation h_0 of the zero set in Proposition 6.6 (applied with $\delta = \sigma$) behaves well under rotations: take $\tilde{h}_0$ to be the function whose graph is obtained by rotating of the graph of h_0 by R^{-1} (cf. [54, Section 8.2]). For z in the domain of $\tilde{h}_0$, there exists $z' \in B_{3/4}^{n-2}$ such that

$$(\tilde{h}_0(z), z) = R^{-1}(h_0(z'), z').$$

Since $\|(P_{\mathbb{R}^{n-2}} \circ R - I) \circ P_{\mathbb{R}^{n-2}}\| \le C\mathbf{E}_1$ and $|h_0| \le \sigma$, we have $|z' - z| \le C\mathbf{E}_1 + C\sqrt{\mathbf{E}_1}\sigma$. Moreover, we have $\mathrm{Lip}(h_0) \le \sigma$, giving $|h_0(z') - h_0(z)| \le C\sqrt{\mathbf{E}_1}\sigma$. Thus, assuming $\sqrt{\mathbf{E}_1} \le \sigma$,

$$(\tilde{h}_0(z), z) = R^{-1}(h_0(z), z) + O(\sqrt{\mathbf{E}_1}\sigma);$$

recalling (8.4), we see that $R(0, z) = (\sqrt{\mathbf{E}_1} dw(0)[z], z) + O(\mathbf{E}_1|z|)$, so that

$$\tilde{h}_0(z) = h_0(z) - \sqrt{\mathbf{E}_1} dw(0)[z] + O(\sqrt{\mathbf{E}_1}\sigma), \tag{8.7}$$

with an implicit constant $C(n)$. Note that $\tilde{h}_0$ can be taken as a Lipschitz approximation of the zero set of the tilted pair: in order to have the conclusion of Proposition 6.6, the only property that we care about is that its graph covers the zeros of $\tilde{u}$, except some exceptional ones projecting on a set of measure at most $C(n)\frac{\mathbf{E}_1}{\eta^2}$, and this holds for the rotated graph.

8.3. **Proof of the excess decay in the general case.** Now we can use (8.7) and the L^2 bound from Proposition 6.6 to conclude the proof of the tilt-excess decay theorem in the general case.

Proof of Theorem 1.4. Recall that $\mathbb{R}^{n-2}$ minimizes $\mathbf{E}_1(u, \nabla, B_1^n, \cdot)$. Let $\tilde{h}$ be the Lipschitz approximation of the barycenter (built in Proposition 6.4) for the tilted pair $(\tilde{u}, \tilde{\nabla})$. We have

$$\begin{aligned}
&|\tilde{h}(z) - (h(z) - \sqrt{\mathbf{E}_1} dw(0)[z])| \\
&\le |\tilde{h} - \tilde{h}_0| + |h - h_0| + |\tilde{h}_0 - (h_0 - \sqrt{\mathbf{E}_1} dw(0)[z])|.
\end{aligned}$$

We combine the main estimate of Proposition 6.6 and (8.6)–(8.7) to see that

$$\begin{aligned}\int_{B^{n-2}_{1/2}} &|\tilde{h}(z) - (h(z) - {\mathbf{E}_1}^{1/2} dw(0)[z])|^2 \\ &\le C\left(\frac{\sigma^2}{\eta^2} + \sigma^2\right)\mathbf{E}_1 + C\varepsilon^2|\log \mathbf{E}|^2\mathbf{E} + Ce^{-K/\varepsilon}.\end{aligned} \tag{8.8}$$

We assume in the sequel that

$$\varepsilon^2|\log \mathbf{E}|^2\mathbf{E}, e^{-K/\varepsilon} \le \sigma^2\mathbf{E}_1,$$

so that

$$\int_{B^{n-2}_{1/2}} |\tilde{h}(z) - (h(z) - \mathbf{E}_1^{1/2} dw(0)[z])|^2 \le C\left(\frac{\sigma^2}{\eta^2} + \sigma^2\right)\mathbf{E}_1.$$

Now, taking the harmonic approximation for the tilted pair to be $\tilde{w}$, we can see that

$$\begin{aligned}&\int_{B^{n-2}_{1/2}} |\tilde{\mathbf{E}}_1^{1/2}\tilde{w}(z) - \mathbf{E}_1^{1/2}(w(z) - dw(0)[z])|^2\,dz \\ &\le C\int_{B^{n-2}_{1/2}} [|h - \mathbf{E}_1^{1/2}w|^2 + |\tilde{h} - \tilde{\mathbf{E}}_1^{1/2}\tilde{w}|^2 + |\tilde{h}(z) - (h(z) - \mathbf{E}_1^{1/2}dw(0)[z])|^2] \\ &\le C\left(\nu + \frac{\sigma^2}{\eta^2} + \sigma^2\right)\mathbf{E}_1\end{aligned}$$

(the last line follows from $\tilde{\mathbf{E}}_1 \le C\mathbf{E}_1$, as we saw in (8.6)). Since

$$\tilde{\mathbf{E}}_1^{1/2}\tilde{w}(z) - \mathbf{E}_1^{1/2}(w(z) - dw(0)[z])$$

is harmonic, its differential at the origin

$$|\tilde{\mathbf{E}}_1^{1/2} d\tilde{w}(0)|^2 \le C\left(\nu + \frac{\sigma^2}{\eta^2} + \sigma^2\right)\mathbf{E}_1.$$

Since $\tilde{\mathbf{E}}_1 \ge \mathbf{E}_1$, this tells us that $|d\tilde{w}(0)|$ can be made arbitrarily small, reducing to the previous situation. □

Remark 8.3. In all the results obtained so far we were assuming that the center of the ball (or cylinder) belongs to the zero set, but actually they also hold if it belongs to the vorticity set $Z = \{|u| \le \frac{3}{4}\}$, since this is enough to guarantee that it belongs to the support of the energy concentration measure in compactness arguments.

9. Iteration arguments and Morrey-type bounds

9.1. **Proof of Theorem 1.9: the case of critical pairs for $2 \le n \le 4$.** We prove the following theorem, which is the first part of Theorem 1.9.

Theorem 9.1. *For $2 \le n \le 4$, there exists $\tau_0(n) > 0$ such that the following holds. If (u, ∇) is an entire critical point for the energy E_1, given by (3.1) for $\varepsilon = 1$, with $u(0) = 0$ and the energy bound*

$$\lim_{R\to\infty} \frac{1}{|B_R^{n-2}|} \int_{B_R^n} \left[|\nabla u|^2 + |F_\nabla|^2 + \frac{1}{4}(1 - |u|^2)^2 \right] \le 2\pi + \tau_0, \tag{9.1}$$

then (u, ∇) is two-dimensional. More precisely, we have $(u, \nabla) = P^*(u_0, \nabla_0)$ up to a change of gauge, where P is the orthogonal projection onto a two-dimensional subspace and (u_0, ∇_0) is the standard degree-one solution of Taubes [51] (or its conjugate), centered at the origin.

Proof. We can assume $n \in \{3, 4\}$. First, we claim that it is enough to show that

$$\lim_{R\to\infty} R^2 \min_S \mathbf{E}_1(u, \nabla, B_R^n, S) = 0.$$

Indeed, once this is done, we have

$$R^{4-n} \int_{B_R^n} \left[\sum_{a=3}^n |\nabla_{e_a^R} u|^2 + \sum_{(a,b)\neq(1,2)} \omega(e_a^R, e_b^R)^2 \right] \to 0$$

as $R \to \infty$, for a suitable choice of planes $S(R)$, where $\{e_1^R, \dots, e_n^R\}$ is an orthonormal basis such that $S(R)$ is spanned by $\{e_3^R, \dots, e_n^R\}$. Extracting a limit $S(R) \to S$ along a subsequence and assuming without loss of generality that $S = \mathbb{R}^{n-2}$, the fact that $n \le 4$ and Fatou's lemma give

$$\int_{\mathbb{R}^n} \left[\sum_{a=3}^n |\nabla_{e_a} u|^2 + \sum_{(a,b)\neq(1,2)} \omega(e_a, e_b)^2 \right] = 0.$$

As in the proof of Lemma 5.6, this implies that (u, ∇) depends only on the first two coordinates up to a change of gauge, and the conclusion follows from the classification of planar solutions by Taubes [51].

We now turn to the previous claim. By Proposition 5.3, we have

$$\frac{1}{|B_R^{n-2}|} \int_{B_R^n} e_\varepsilon(u, \nabla) \to 2\pi$$

as $R \to \infty$, as well as

$$\mathbf{E}(u, \nabla, B_R^n, S(R)) \to 0$$

for suitable oriented planes $S(R)$, up to conjugating the pair. Arguing as in the proof of Proposition 5.3, we see that $S(R)$, viewed as an unoriented plane, has vanishing distance from any unoriented plane S minimizing $\mathbf{E}_1(u, \nabla, B_R^n, S)$; hence, we can assume that $S(R)$ minimizes $\mathbf{E}_1$ on B_R^n.

The proof now becomes an elementary iteration argument. In Theorem 1.4 we first fix $\rho \in (0, 1)$ such that $C\rho^2 \le \rho$ and then τ and ε_0 accordingly. Let $C' > \frac{1}{\varepsilon_0}$. Without loss of generality we can also assume that

$$\mathbf{E}_1(u, \nabla, B_R^n, S(R)) > 0, \quad \mathbf{E}(u, \nabla, B_R^n, S(R)) \in (0, 1)$$

are small enough to allow applying Theorem 1.4 on B_R^n (by rescaling our pair), for all $R \geq C'$. For every $k \in \mathbb{N}$ let us define the minimum excess on each ball $B_{C'\rho^{-k}}$:

$$\mathbf{E}_1(k) := \mathbf{E}_1(u, \nabla, B_{C'\rho^{-k}}, S(C'\rho^{-k})).$$

Then, up to changing $K > 0$, Theorem 1.4 gives

$$\begin{aligned} &\text{either } \mathbf{E}_1(k) \leq \rho \mathbf{E}_1(k+1) \\ &\quad \text{or } \mathbf{E}_1(k) \leq \max\{\rho^{2k} |\log \mathbf{E}(k+1)|^2 \sqrt{\mathbf{E}(k+1)}, e^{-K\rho^{-k}}\}, \end{aligned} \tag{9.2}$$

where $\mathbf{E}(k) := \mathbf{E}(u, \nabla, B_{C'\rho^{-k}}, S(C'\rho^{-k}))$. By Proposition 5.3, we have

$$\lim_{k\to\infty} \mathbf{E}_1(k) = 0, \quad \lim_{k\to\infty} \mathbf{E}(k) = 0. \tag{9.3}$$

In order to iterate (9.2), we define

$$f(k) := \log \mathbf{E}_1(k) + 2k \log \rho^{-1}$$

and

$$g(k) := \max\left\{2 \log |\log \mathbf{E}(k+1)| + \frac{1}{2} \log \mathbf{E}(k+1), -K\rho^{-k} + 2k \log \rho^{-1}\right\}.$$

Then (9.2) can be rewritten in terms of the functions $f, g : \mathbb{N} \to \mathbb{R}$ as

$$f(k) \leq f(k+1) - \lambda \quad \text{or} \quad f(k) \leq g(k), \tag{9.4}$$

where $\lambda := 3 \log \rho^{-1}$. Condition (9.3) also means that

$$\lim_{k\to\infty} f(k) - 2k \log \rho^{-1} = -\infty, \quad \lim_{k\to\infty} g(k) = -\infty. \tag{9.5}$$

We claim that if f, g satisfy (9.4) and (9.5) then

$$f(k) \leq \sup_{m \geq k} [g(m) - \lambda(m-k)].$$

We prove this by contradiction: assume that there is some index k_0 such that

$$f(k_0) + \lambda(m - k_0) > g(m) \quad \text{for all } m \geq k_0. \tag{9.6}$$

In particular we have $f(k_0) > g(k_0)$, so that (9.4) and (9.6) give

$$f(k_0 + 1) \geq f(k_0) + \lambda > g(k_0 + 1).$$

By induction, we see that for all $m \geq k_0$

$$f(m) \geq f(k_0) + \lambda(m - k_0).$$

Taking the limit $m \to \infty$ and noting that $\lambda > 2 \log \rho^{-1}$, we obtain

$$\begin{aligned} f(k_0) &\leq \lim_{m\to\infty} [f(m) - \lambda(m - k_0)] \\ &\leq \lim_{m\to\infty} [f(m) - 2m \log \rho^{-1}] + 2k_0 \log \rho^{-1} \\ &= -\infty, \end{aligned}$$

where we used (9.5) in the last equality. This is a contradiction, proving our claim.

As a consequence, we have

$$f(k) \le \sup_{m\ge k}[g(m) - \lambda(m-k)] \le \sup_{m\ge k} g(m).$$

Since $\lim_{k\to\infty} g(k) = -\infty$ by (9.5), we deduce that

$$\lim_{k\to\infty} f(k) = -\infty.$$

In other words, we have $\rho^{-2k}\mathbf{E}_1(k) \to 0$, as desired. □

9.2. **Proof of Corollary 1.6 and Theorem 1.7.** Given any $n \ge 3$ and (u,∇) as in Theorem 1.7, for any $\tau_0' > 0$ a standard compactness argument shows that

$$\frac{1}{|B_r^{n-2}|}\int_{B_r(x)} e_\varepsilon(u,\nabla) \le 2\pi + \tau_0'$$

for all $x \in Z \cap B^n_{3/4}$ and $r = \frac{1}{8}$, provided that τ_0 and ε_0 are taken small enough, and hence also for $r \le \frac{1}{7}$ by energy monotonicity.

This, together with Proposition 5.3, implies that, for some oriented planes $S(x,r)$,

$$\mathbf{E}(u,\nabla,B_r(x),S(x,r)) \le \delta$$

for some $\delta > 0$ to be chosen momentarily and $C(n,\delta)\varepsilon \le r \le \frac{1}{8}$. As in the previous proof, we can assume that $S(x,r)$ minimizes $\mathbf{E}_1$ on the ball $B_r(x)$.

Given $\alpha \in [0,1)$, we first fix ρ such that $C\rho^2 \le \rho^{2\alpha}$ where C is the constant appearing in Theorem 1.4. We now choose δ, τ_0 small such that Theorem 1.4 applies on each ball $B_r(x)$ with $x \in Z \cap B^n_{3/4}$, compare with Remark 8.3. We then consider

$$\max\{M\varepsilon, \varepsilon^{1/(1+\alpha)}\} \le r \le \frac{1}{8} \tag{9.7}$$

where M chosen large enough to ensure that

$$e^{-Kr/\varepsilon} \le \frac{\varepsilon^2}{r^2}$$

if $\varepsilon/r \le 1/M$. Applying the scaled version of Theorem 1.4 (with ε replaced by ε/r), and noticing that $\sup_{0<s\le\delta}|\log s|^2 s^{1/2} \le 1$, we finally obtain that either

$$\mathbf{E}_1(u,\nabla,B_{\rho r}(x),S(x,\rho r)) \le \rho^{2\alpha}\mathbf{E}_1(u,\nabla,B_r(x),S(x,r))$$

or

$$\mathbf{E}_1(x,r) := \mathbf{E}_1(u,\nabla,B_r(x),S(x,r)) \le \frac{\varepsilon^2}{r^2} \le r^{2\alpha}.$$

This immediately implies

$$\mathbf{E}_1(x,r) \le C(n,\alpha)r^{2\alpha} \quad \text{for all } x \in Z \cap B^n_{3/4} \text{ and any radii satisfying (9.7).}$$

Moreover, if $S(x,r)$ is different from $S(x,r')$, for some $r' \in [r,2r]$, then we can find an orthonormal basis $\{e_1,\dots,e_n\}$ such that $\{e_3,\dots,e_n\}$ spans $S(x,r)$

and e_2 belongs to the span of the two planes, namely $e_2 = v + w$ with $v \in S(x, r)$ and $w \in S(x, r')$, with the bound

$$|v| + |w| \le C(n) \| P_{S(x,r)} - P_{S(x,r')} \|^{-1},$$

as the next simple lemma shows.

Lemma 9.2. *Given two different planes $S, S' \in \mathrm{Gr}(n, k)$, there exists a unit vector $e \in (S + S') \cap S^\perp$ such that $e = v + w$, with $v \in S$, $w \in S'$, and $|v|, |w| \le C(n) \| P_S - P_{S'} \|^{-1}$.*

Proof. We can assume that $S + S' = \mathbb{R}^n$ and $S \cap S' = \{0\}$ (otherwise we work on $(S \cap S')^\perp$), so that $n = 2k$. We can also assume without loss of generality that $\| P_S - P_{S'} \|_{op} < c(n)$ for a constant $c(n) > 0$ to be determined momentarily, since otherwise the statement follows from an immediate compactness argument (using the fact that, if $S_j \to S_\infty$ and $S'_j \to S'_\infty$, then each unit vector in $S_\infty + S'_\infty$ has vanishing distance from $S_j + S'_j$, even when the former sum has smaller dimension).

It is elementary to check that the statement holds when $k = 1$: in this case, calling $\theta \in (0, \frac{\pi}{2}]$ the angle between the lines S and S', we have $\| P_S - P_{S'} \| = \sqrt{2} \sin \theta$, and we can find vectors as in the statement with $|v|, |w| \le \frac{1}{\sin \theta}$.

Let $\tilde{e}$ be an eigenvector of $P_S - P_{S'}$, corresponding to an eigenvalue λ with $0 < |\lambda| = \| P_S - P_{S'} \|_{op} < c(n)$. Then

$$P_S \tilde{e} - P_{S'} \tilde{e} = \lambda \tilde{e},$$

so that in particular $P_S \tilde{e}, P_{S'} \tilde{e} \ne 0$ and

$$P_S P_{S'} \tilde{e} = (1 - \lambda) P_S \tilde{e}.$$

Similarly we have

$$P_{S'} P_S \tilde{e} = (1 + \lambda) P_{S'} \tilde{e}.$$

From the equation

$$\langle P_{S'} P_S \tilde{e}, P_{S'} \tilde{e} \rangle = \langle P_S \tilde{e}, P_{S'} \tilde{e} \rangle = \langle P_S \tilde{e}, P_S P_{S'} \tilde{e} \rangle$$

we deduce that

$$|P_{S'} \tilde{e}|^2 = \frac{1 - \lambda}{1 + \lambda} |P_S \tilde{e}|^2.$$

In particular, calling $\theta \in (0, \frac{\pi}{2}]$ the angle between the vectors $P_S \tilde{e}$ and $P_{S'} \tilde{e}$, these identities easily give

$$\sin^2 \theta = 1 - \frac{\langle P_S \tilde{e}, P_{S'} \tilde{e} \rangle^2}{|P_S \tilde{e}|^2 |P_{S'} \tilde{e}|^2} = \lambda^2.$$

We now take

$$Z := \mathrm{span}\{P_S \tilde{e}, P_{S'} \tilde{e}\},$$

which is a two-dimensional plane. By the case $k = 1$, we can find

$$e \in Z, \quad v \in \mathrm{span}\{P_S \tilde{e}\}, \quad w \in \mathrm{span}\{P_{S'} \tilde{e}\}$$

such that $e \perp P_S\tilde{e}$ is a unit vector and $|v|, |w| \le \lambda^{-1}$. In order to conclude, it suffices to check that $e \perp S$. Writing

$$e = \alpha P_S\tilde{e} + \beta P_{S'}\tilde{e},$$

we have

$$P_S e = \alpha P_S\tilde{e} + \beta P_S P_{S'}\tilde{e} = [\alpha + \beta(1-\lambda)]P_S\tilde{e}.$$

Since $e \perp P_S\tilde{e}$, we have

$$0 = \langle \alpha P_S\tilde{e} + \beta P_{S'}\tilde{e}, P_S\tilde{e}\rangle = \alpha|P_S\tilde{e}|^2 + \beta\langle P_S P_{S'}\tilde{e}, P_S\tilde{e}\rangle = [\alpha+\beta(1-\lambda)]|P_S\tilde{e}|^2.$$

Since $P_S\tilde{e} \ne 0$, we have $\alpha + \beta(1-\lambda) = 0$, proving the claim. □

Since $\mathbf{E}(x,r) \le \delta$, we have

$$\begin{aligned} &r^{2-n}\int_{B_r(x)} e_\varepsilon(u,\nabla) \\ &\le C(n)\delta + C(n)r^{2-n}\int_{B_r(x)}\left[\sum_{k=2}^n |\nabla_{e_k}u|^2 + \sum_{(j,k)} \varepsilon^2\omega(e_j,e_k)^2\right]. \end{aligned}$$

Since the left-hand side is close to 2π, and in particular larger than π (for $r \ge C\varepsilon$), using the previous fact from linear algebra for the term $\nabla_{e_2}u$ we obtain

$$1 \le C(n)[\mathbf{E}_1(x,r) + \mathbf{E}_1(x,r')](\|P_{S(x,r)} - P_{S(x,r')}\|^{-2} + 1),$$

and thus, since $\|P_{S(x,r)} - P_{S(x,r')}\| \le C(n)$,

$$\|P_{S(x,r)} - P_{S(x,r')}\| \le C(n)\sqrt{\mathbf{E}_1(x,r) + \mathbf{E}_1(x,r')} \le C(n,\alpha)r^\alpha. \tag{9.8}$$

As a consequence, summing over dyadic scales, we have

$$\|P_{S(x,r)} - P_{S(x,s)}\| \le C(n,\alpha)\max\{r,s\}^\alpha$$

for $\max\{C(n,\alpha)\varepsilon, \varepsilon^{1/(1+\alpha)}\} \le r,s \le \frac{1}{8}$.

A similar argument works varying the center: for two different points $x,x' \in Z \cap B^n_{3/4}$, looking at the balls $B_r(x) \subset B_{2r}(x')$ with $r := |x-x'|$, we also have

$$\|P_{S(x,r)} - P_{S(x',r)}\| \le C(n,\alpha)r^\alpha, \tag{9.9}$$

provided that $r = |x-x'| \in [\max\{C(n,\alpha)\varepsilon, \varepsilon^{1/(1+\alpha)}\}, \frac{1}{16}]$.

Actually, the previous proof gives some extra information, which will be crucial in the sequel. We record it in the next proposition.

Proposition 9.3. *Up to a rotation, we have* $\|P_{S(x,r)} - P_{\mathbb{R}^{n-2}}\| \le \gamma$ *for any* $\gamma > 0$ *fixed in advance (up to decreasing* ε_0, τ_0*), for all* $x \in Z \cap B^n_{3/4}$ *and* $r \in [C(n,\gamma)\varepsilon, \frac{1}{8}]$.

Proof. Let C be the constant in the excess decay statement, fix ρ such that $C\rho^2 \leq \rho$ and fix τ_0 and ε_0 accordingly. Letting $r_k := \rho^k$, the first inequality of (9.8) gives that

$$\|P_{k+1} - P_k\| \leq C\sqrt{\mathbf{E}_1(x, r_k)}.$$

Iterating we get that

$$\begin{aligned}\|P_\ell - P_k\| &\leq C(n)\sum_{j=k}^{\ell-1}\sqrt{\mathbf{E}_1(x, r_j)}\\ &\leq C(n)\sqrt{\mathbf{E}_1(x, r_k)}(1 + \rho^{1/2} + \rho + \dots)\\ &\leq C\sqrt{\mathbf{E}_1(x, r_k)}\end{aligned}$$

as long as $\mathbf{E}_1(x, r_j) > \frac{\varepsilon^2}{r_j^2}$ for $j = 0, \dots, \ell - 1$ and $r_\ell \geq M\varepsilon =: \overline{r}$ where M is a large constant that we will fix at the end. Hence, if we call $r_{k_1} > \dots > r_{k_N} \geq \overline{r}$ the possible radii where $\mathbf{E}_1(x, r_{k_i}) \leq \frac{\varepsilon^2}{r_{k_i}^2}$, we deduce that

$$\begin{aligned}\|P_\ell - P_0\| &\leq C\max\{\sqrt{\mathbf{E}_1(x, r_0)}, \sqrt{\mathbf{E}_1(x, r_{k_1})}, \dots, \sqrt{\mathbf{E}_1(x, r_{k_N})}\}\\ &\leq C\left[\sqrt{\mathbf{E}_1(x, r_0)} + \frac{\varepsilon}{\overline{r}}\right]\\ &\leq C\left[\sqrt{\mathbf{E}_1(x, r_0)} + \frac{1}{M}\right].\end{aligned}$$

Also, P_0 can be assumed arbitrarily close to $\mathbb{R}^{n-2}$ by a simple compactness argument (similar to the proof of Proposition 5.3). Since $\sqrt{\mathbf{E}_1(x, r_0)}$ and $1/M$ can be taken arbitrarily small, the claim follows. □

The same proof gives the following.

Proposition 9.4. *For any $x \in Z \cap B^n_{3/4}$ and $r \in [C(n)\varepsilon, \frac{1}{8}]$, we have*

$$\mathbf{E}_1(u, \nabla, B_r(x), \mathbb{R}^{n-2}) \leq C(n)\mathbf{E}_1(u, \nabla, B_1(0), \mathbb{R}^{n-2}) + C(n)\frac{\varepsilon^2}{r^2}.$$

We now prove Corollary 1.6.

Proof of Corollary 1.6. We have already seen in Proposition 5.3 that the energy on B_R is asymptotic to $2\pi R^{n-2}$. We can then apply Proposition 9.3: for any $\gamma > 0$ we have

$$\|P_{S(0,R)} - P_{S(0,R')}\| \leq \gamma$$

for $R < R'$ large enough (we use Proposition 9.3 after scaling the picture down by a factor $(R')^{-1}$). We deduce that the limit

$$\lim_{R\to\infty} S(0, R)$$

exists. □

Proposition 9.5. *Up to a rotation, the vorticity set* $\tilde{Z} := Z \cap [B^2_{1/2} \times B^{n-2}_{1/2}]$ *is included in a* $C(n,\gamma)\varepsilon$*-neighborhood of the graph of a* C^1 *map*

$$f : B^{n-2}_{1/2} \to B^2_\gamma$$

with $\mathrm{Lip}(f) \le \gamma$*, if we assume that* τ_0 *and* ε_0 *are small enough (depending on* n, γ*).*

Proof. Indeed, as seen in the proof of Proposition 5.3, for ε small enough we have $u(\cdot, z) \neq 0$ on $\partial B^2_{1/2}$, for all $z \in B^{n-2}_{1/2}$, and the degree of $(u/|u|)(\cdot, z)$ is ± 1 on this circle. Hence, each slice $B^2_{1/2} \times \{z\}$ intersects the zero set.

Moreover, using Lemma 5.4 on $B_1(0)$, we see that $\tilde{Z} \subseteq B^2_\gamma \times B^{n-2}_{1/2}$. Also, Lemma 5.4 implies that for all $x \in Z \cap B^n_{3/4}$ and $r \in [C(n,\gamma)\varepsilon, \frac{1}{8}]$ we have

$$Z \cap B_r(x) \subseteq B_{\gamma r}(x + S(x,r)), \tag{9.10}$$

where $B_{\gamma r}(x + S(x,r))$ is the γr-neighborhood of the affine plane $x + S(x,r)$. We now take a collection of points $\{z_k\} \subset B^{n-2}_{1/2}$ with pairwise distance at least $C(n,\gamma)\varepsilon$ and $B^{n-2}_{1/2} \subseteq \bigcup_k B^{n-2}_{5C(n,\gamma)\varepsilon}(z_k)$. For each k, we fix a point $x_k = (y_k, z_k) \in \tilde{Z}$. We then see that

$$|y_k - y_j| \le C\gamma |x_k - x_j|,$$

thanks to the previous observation applied with $r := 2|x_k - x_j|$ and the fact that $S(x,r)$ is γ-close to $\mathbb{R}^{n-2}$ (for $|x_k - x_j| > \frac{1}{16}$, this follows just from Lemma 5.4). Hence, the assignment $z_k \mapsto y_k$ defines a $C(n)\gamma$-Lipschitz function, which we can extend to a $C(n)\gamma$-Lipschitz function $f : B^{n-2}_{1/2} \to B^2_\gamma$. It is easy to check that (a regularization of) f satisfies the desired conclusion, completing the proof. □

We are now in position to prove Theorem 1.7.

Proof of Theorem 1.7. Let $\eta > 0$ small such that Proposition 6.4 applies. We first remark that the previous points x_k can be taken such that $u(x_k) = 0$ and $z_k \in \mathcal{G}^\eta$. Indeed, by Proposition 9.4, we have

$$\mathbf{E}_1(u, \nabla, B_r(x), \mathbb{R}^{n-2}) \le c(n)\eta^2$$

for all points $x \in Z \cap B^n_{3/4}$ and radii $r \in [C(n)\varepsilon, \frac{1}{8}]$ (by taking ε_0, τ_0 suitably small). We can apply this with $r := M\varepsilon$; by Proposition 9.5 and exponential decay of energy away from Z, we have

$$r^{2-n} \int_{B^{n-2}_{r/2}(z)} (\mathbf{E}_1)_z \le c(n)\eta^2 + e^{-K/M} \le 2c(n)\eta^2$$

once we take $M = C(n)$ large enough, for any $z \in B^{n-2}_{1/2}$. Once we take $c(n)$ small enough, by the weak L^1 bound we can then find

$$z' \in B^{n-2}_{r/2}(z) \cap \mathcal{G}^\eta$$

(where we use slices of radius $\frac{1}{2}$ in the definition of $\mathcal{G}^\eta$), showing the claim.

As a consequence of Lemma A.1, we deduce that

$$|h(z_k) - y_k| = |h(z_k) - h_0(z_k)| \le \varepsilon.$$

We immediately deduce that $\tilde{Z}$ is included in a $C(n)\varepsilon$-neighborhood of the graph of h, which is the only consequence of the claim which we will use in the sequel.

Now let $\overline{\rho} := \max\{M\varepsilon, \varepsilon^{1/(1+\alpha)}\}$ (with M as in (9.7)) and consider *another* finite collection of points $\{z_k\} \subset B^{n-2}_{1/2}$ such that the balls $B^{n-2}_{\overline{\rho}}(z_k)$ are disjoint and the dilated balls $B^{n-2}_{4\overline{\rho}}(z_k)$ cover $B^{n-2}_{1/2}$. Let $x_k = (y_k, z_k)$ be a point in $\tilde{Z}$ for each k.

On $B^n_{10\overline{\rho}}(x_k)$ we consider the Lipschitz approximation h_k built with respect to the rotated picture, obtained as a graph over $S_k := x_k + S(x_k, 10\overline{\rho})$. When viewed as a graph over $\mathbb{R}^{n-2}$, it becomes a function $\tilde{h}_k$ defined on the slightly smaller ball $B^{n-2}_{5\overline{\rho}}(z_k)$.

By a scaled version of Proposition 6.4, we have

$$\int_{B^n_{30\overline{\rho}/4}(x_k)\cap S_k} |dh_k|^2 \le C\overline{\rho}^{n-2}\mathbf{E}_1(u, \nabla, B^n_{10\overline{\rho}}(x_k), S(x_k, 10\overline{\rho})) \le C\overline{\rho}^{n-2+2\alpha}.$$

In particular, by Poincaré,

$$\int_{B^n_{30\overline{\rho}/4}(x_k)\cap S_k} |h_k - (h_k)|^2 \le C\overline{\rho}^{n+2\alpha}.$$

Now, as in (8.7), we observe that

$$|\tilde{h}_k(z) - h_k(z) - A_k(z)| \le C\overline{\rho}\sqrt{\mathbf{E}_1(u, \nabla, B^n_{10\overline{\rho}}(x_k), S(x_k, 10\overline{\rho}))} \le C\overline{\rho}^{1+\alpha}$$

for a suitable affine function A_k (where, with abuse of notation, $h_k(z)$ means h_k composed with the isometry $\mathbb{R}^{n-2} \to S_k$). Combining these two bounds, we get

$$\int_{B^{n-2}_{5\overline{\rho}}(z_k)} |\tilde{h}_k - A'_k|^2 \le C\overline{\rho}^{n+2\alpha}$$

for another affine function A'_k.

We now take a partition of unity φ_k subordinated to the cover $\{B^{n-2}_{4\overline{\rho}}(z_k)\}$ and we let

$$f := \sum_k \varphi_k \tilde{h}_k.$$

Since the zero set is within a $C\varepsilon$-neighborhood of the graph of $\tilde{h}_k$ (on the set $B^2_{1/2} \times B^{n-2}_{5\overline{\rho}}(z_k)$), we deduce that

$$|\tilde{h}_k - \tilde{h}_{k'}| \le C\varepsilon$$

on $B^{n-2}_{5\overline{\rho}}(z_k) \cap B^{n-2}_{5\overline{\rho}}(z_{k'})$. Thus, we also have

$$|A_k - A_{k'}| \le C\varepsilon + C\overline{\rho}^{1+\alpha}$$

whenever $B^{n-2}_{4\overline{\rho}}(z_k) \cap B^{n-2}_{4\overline{\rho}}(z_{k'}) \neq \emptyset$. Since $\varepsilon \leq \overline{\rho}^{1+\alpha}$, this allows us to conclude that

$$\int_{B^{n-2}_{2\overline{\rho}}(z)} |f - A_z|^2 \leq C\overline{\rho}^{n+2\alpha}$$

for any $z \in B^{n-2}_{1/2}$, for a suitable affine function A_z depending on z.

Thus, taking a standard mollifier $\chi_{\overline{\rho}}$, setting

$$g := \chi_{\overline{\rho}} * f$$

and using the previous bound, we deduce that

$$|dg - dA_z| \leq C\overline{\rho}^{\alpha} \quad \text{on } B^{n-2}_{\overline{\rho}}(z),$$

and in fact

$$[dg]_{C^{0,\alpha}(B^{n-2}_{\overline{\rho}}(z))} \leq C.$$

Finally, recalling that dA_k is the slope of the plane $P_{S(x_k,10\overline{\rho})}$, we also have

$$|dA_k - dA_{k'}| \leq C|z_k - z_{k'}|^{\alpha}$$

by the Hölder continuity (9.9), while

$$|dA_z - dA_k| \leq C\overline{\rho}^{\alpha} \quad \text{for } z \in B^{n-2}_{4\overline{\rho}}(z_k).$$

From these bounds, we easily deduce that

$$|dg(z) - dg(z')| \leq C|z - z'|^{\alpha} \quad \text{for } |z - z'| \geq \overline{\rho},$$

completing the proof of the $C^{1,\alpha}$ regularity of g. Since $|g - f| \leq C\overline{\rho}$, it follows that the vorticity set is included in a $C\overline{\rho}$-neighborhood of the graph of g.

It is clear from the proof that we can actually make $[dg]_{C^{0,\alpha}}$ arbitrarily small, up to decreasing τ_0 and ε_0. □

10. Constructing competitors for local minimizers: a good gauge

In this section we prepare the ground to construct competitors for minimizers and to show that the full excess decays as long as it is above ε^{β}, for any $\beta > 0$, giving a proof of Theorem 1.10. To investigate minimizers, we construct competitors with the same boundary conditions and compare the energies to show that the excess $\mathbf{E}$ is effectively approximated by the Dirichlet energy of the harmonic approximation.

To do this, we need to construct competitors modeled on graphs in the interior and then glue them to the boundary condition, while controlling the error terms. We pullback the ε-rescaled degree-one solution along the graph of the Lipschitz approximation, as obtained in Proposition 6.4. Then we gauge fix in balls of size $\varepsilon|\log\varepsilon|$ and, using the estimates at that scale, we define a global gauge by a partition of unity. In this gauge we can interpolate between the intial pair and the new one with good estimates.

10.1. **The pullback pair.** Here we introduce the pullback pair (u_f, ∇_f) whose zero set is prescribed to be the graph of a Lipschitz function $f : B^{n-2} \to B_1^2$. We prove that the excess of these pairs are well approximated by the Dirichlet energy of f, as in the following proposition.

Proposition 10.1 (Constructing the pullback pair)**.** *There exist small constants $\eta_0(n), \varepsilon_0(n) > 0$ with the following property. Given any $\varepsilon \le \varepsilon_0$ and a Lipschitz function $f : B_1^{n-2} \to B_{1/2}^2$ with $Lip(f) = \eta \le \eta_0$, there exists a pair (u_f, ∇_f) obeying the following estimate:*

$$\frac{1}{2\pi}\int_{B_1^2\times B_1^{n-2}} e_\varepsilon(u_f,\nabla_f) = |B_1^{n-2}| + (1+O(\eta^2))\int_{B_1^{n-2}}\frac{|df|^2}{2} + O(e^{-K/\varepsilon}),$$

with implicit constants $C(n)$. Moreover, we have that

$$u_f^{-1}(0) = \mathrm{graph}(f).$$

Proof. To construct (u_f, ∇_f) we pull back the planar degree-one solution of Taubes [51], via the map $Q_\varepsilon : B_1^2 \times B_1^{n-2} \to \mathbb{R}^2$ given by

$$Q_\varepsilon(x) = \frac{(x_1,x_2) - f(x_3,\dots,x_n)}{\varepsilon}.$$

Then we define (u_f, ∇_f) by

$$(u_f,\nabla_f) := Q_\varepsilon^*(u_0,\nabla_0), \tag{10.1}$$

where (u_0, ∇_0) is the degree-one solution in [51] with $u_0(0) = 0$ (unique up to change of gauge). First, we note that, since

$$dQ_\varepsilon(x)[e_k] = -\partial_{e_k} f_1(x_3,\dots,x_n)e_1 - \partial_{e_k} f_2(x_3,\dots,x_n)e_2,$$

we have the following identities for $k = 3, \dots, n$:

$$\begin{aligned} |(\nabla_f)_k u_f|^2(x) &= \varepsilon^{-2}|\partial_{e_k} f_1(\nabla_0)_{e_1}u_0 + \partial_{e_k} f_2(\nabla_0)_{e_2}u_0|^2\left(Q_\varepsilon(x)\right) \\ &= \varepsilon^{-2}\frac{|\partial_k f|^2}{2}|\nabla_0 u_0|^2\left(Q_\varepsilon(x)\right), \end{aligned} \tag{10.2}$$

where we omitted the argument of f and we used the fact that $(\nabla_0)_{e_2}u_0 = i(\nabla_0)_{e_1}u_0$ for solutions of the vortex equations (4.13). We also have

$$|(\nabla_f)_{e_1}u_f|^2(x) + |(\nabla_f)_{e_2}u_f|^2(x) = \varepsilon^{-2}|\nabla_0 u_0|^2\left(Q_\varepsilon(x)\right). \tag{10.3}$$

We also compute for the curvature term $-i\omega_f := F_{\nabla_f} = F_{Q_\varepsilon^*(\nabla_0)} = Q_\varepsilon^*(F_{\nabla_0})$ that

$$\sum_{j=1,2}\varepsilon^2\omega_f(e_j,e_k)^2(x) = \varepsilon^{-2}\omega_0(e_1,e_2)^2\left(Q_\varepsilon(x)\right)|\partial_{e_j} f|^2 \quad \text{for } j=1,2,\ k\ge 3, \tag{10.4}$$

and moreover

$$\varepsilon^2\omega_f(e_1,e_2)^2(x) = \varepsilon^{-2}\omega_0^2(e_1,e_2)\left(Q_\varepsilon(x)\right), \tag{10.5}$$

as well as

$$\varepsilon^2\omega_f(e_k,e_\ell)^2(x) \le \varepsilon^{-2}|df|^4\omega_0(e_1,e_2)^2\,(Q_\varepsilon(x)) \quad \text{for } 3\le k<\ell\le n. \tag{10.6}$$

To compute $E_\varepsilon(u_f,\nabla_f)$, we use (10.2)–(10.6) to see that

$$\begin{aligned}
&\int_{B_1^2\times B_1^{n-2}} e_\varepsilon(u,\nabla)\\
&= \int_{B_1^{n-2}}\left[\int_{B_1^2}\varepsilon^{-2}e_1(u_0,\nabla_0)(Q_\varepsilon(x))\left(1+\frac{|df|^2}{2}+O(|df|^4)\right)+O(e^{-K/\varepsilon})\right]\\
&= 2\pi\left[|B_1^{n-2}|+\int_{B_1^{n-2}}\frac{|df|^2}{2}+O(|df|^4)\right]+O(e^{-K/\varepsilon}).
\end{aligned}$$

In the above display we used the exponential decay from [36, Chapter III, Theorem 8.1]:

$$\int_{\mathbb{R}^2\setminus B^2_{1/(2\varepsilon)}} e_1(u_0,\nabla_0) = O(e^{-K/\varepsilon}).$$

Recalling that $|df|\le\eta$ we get the desired estimate. □

10.2. **Constructing the interpolation gauge.** In this section we find a gauge transformation $(u,\nabla)\mapsto(e^{i\xi}u,\nabla-id\xi)$ for which the new pair is L^2-close to the pullback pair (u_h,∇_h) constructed in Proposition 10.1, where $h: B_1^{n-2}\to B^2_{1/2}$ is the Lipschitz approximation built in Proposition 6.4. Since this is the most technical part of the paper, we provide an overview of the arguments.

Step 1. To begin with, we cover the vortex set with cylinders of the form $\{B^2_{5C|\varepsilon\log\varepsilon|}(y_k)\times B^{n-2}_{5C\varepsilon}(x_k)\}_{k=1}^N$ with $x_k=(y_k,z_k)\in\mathbb{R}^2\times\mathbb{R}^{n-2}$ such that $B^{n-2}_{5C\varepsilon}(z_k)$ is a Vitali cover of B_1^{n-2}. Then we name a cylinder *good* if the excess on it is small, and *bad* otherwise. We also define a partition of unity, subordinate to this cover, with derivatives at most $C\varepsilon^{-1}$.

Step 2. In each cylinder we pass to the Coulomb gauge, via a function ξ_k with mean equal to the mean of $\theta_h-\theta$ on an appropriate annulus away from the vortex set (note that $\theta-\theta_h$ is well-defined far from the vortex set of both u and u_h) Then we use Gaffney- and Poincaré-type inequalities from Appendix B to derive estimates for $e^{i\xi_k}u-u_h$ and $(\alpha+d\xi_k)-\alpha_h$, where we write $\nabla=d-i\alpha$. Far from the vorticity set we modify the gauge so that $e^{i\xi_k}u$ and u_h have the same phase. Then we use the exponential decay away from the vortex set, which is where the error ε^β comes from; however, this will be enough to show the classification result in all dimensions, since we are free to take β arbitrarily large.

Step 3. We use the estimates on $(\alpha+d\xi_k)-\alpha_h$ (and the mean condition) and Poincaré–Gaffney-type inequalities from Appendix B to bound $\xi_j-\xi_k$ on overlapping cylinders.

Step 4. We patch together the ξ_k's with the partition of unity defined in the first step to obtain the function ξ. Then we use the bounds on $\xi_j - \xi_k$ to derive estimates on $(\alpha + d\xi) - \alpha_h$ and $e^{i\xi}u - u_h$.

Proposition 10.2 (The interpolation gauge)**.** *For any $\beta > 0$ there exist $\tau_0(n, \beta), \varepsilon_0(n, \beta) > 0$ and $C_0(n, \beta) > 0$ with the following property. Let (u, ∇) be a critical pair for E_ε on $\mathbb{R}^n$, with $u(0) = 0$, $\varepsilon \le \varepsilon_0$, and the energy bound*

$$\frac{1}{|B_2^{n-2}|}\int_{B_2^n} e_\varepsilon(u, \nabla) \le 2\pi + \tau_0.$$

Moreover, let $h : B_1^{n-2} \to B_{1/2}^2$ be the Lipschitz approximation defined in Proposition 6.4 (for a suitable η chosen later on) and let (u_h, ∇_h) be the pullback pair constructed in Proposition 10.1. Then, on a given annulus

$$\mathcal{A}_{s,\delta} := B_1^2 \times (B_{s+\delta}^{n-2} \setminus \overline{B}_s^{n-2})$$

with $\delta \in [C_0\varepsilon, \frac{1}{16}]$ and $s \le \frac{3}{4}$, we can find a gauge transformation

$$(u, \nabla) \mapsto (e^{i\xi}u, \nabla - id\xi),$$

via a smooth function $\xi : \mathcal{A}_{s,\delta} \to \mathbb{R}$ such that:

(i) letting $P : \mathbb{R}^n \to \mathbb{R}^{n-2}$ be the projection onto the last $n-2$ coordinates, we have

$$\int_{\mathcal{A}_{s,\delta}} [\varepsilon^{-2}|e^{i\xi}u - u_h|^2 + |(\alpha + d\xi) - \alpha_h|^2]$$
$$\le C(n, \beta)|\log\varepsilon|^8 \int_{P(\mathcal{A}_{s-\delta,3\delta})} [\mathbf{E}_z + \mathbf{1}_{\mathcal{G}^\eta}\mathbf{E}_z|\log\mathbf{E}_z|^2] + \varepsilon^\beta;$$

(ii) letting $Z := \{|u| \le 3/4\}$ and

$$Z_{C_0\varepsilon|\log\varepsilon|} := \{x = (y, z) \,:\, \operatorname{dist}(x, Z \cap (B_1^2 \times \{z\})) \le C_0\varepsilon|\log\varepsilon|\},$$

the function $e^{i\xi}u \to \mathbb{C} \setminus \{0\}$ has the same phase as u_h far from the vortex set, i.e.,

$$\frac{e^{i\xi}u}{u_h} \in \mathbb{R}^+ \quad \text{on } \mathcal{A}_{s,\delta} \setminus Z_{5C_0|\varepsilon\log\varepsilon|}.$$

Proof. First we choose ε_0, τ_0 small enough so that Theorem 1.7 applies (for $\alpha = 0$). We divide the proof into several steps.

Covering arguments and a partition of unity. To begin with, by Theorem 1.7, we can find a collection of points

$$\{x_k = (y_k, z_k)\}_{k=1}^N \subset Z \cap \mathcal{A}_{s,\delta}$$

satisfying the following.

(i) The projected collection $\{z_k = P(x_k)\}_{k=1}^N \subset B^{n-2}_{s+\delta} \setminus B^{n-2}_s$ gives a Vitali covering of the projected annulus $P(\mathcal{A}_{s,\delta})$:

$$
\begin{aligned}
&P(\mathcal{A}_{s,\delta}) = B^{n-2}_{s+\delta} \setminus \overline{B}^{n-2}_s \subseteq \bigcup_{k=1}^N B^{n-2}_{5C_0\varepsilon}(z_k), \\
&B^{n-2}_{C_0\varepsilon}(z_j) \cap B^{n-2}_{C_0\varepsilon}(z_k) = \emptyset \quad \text{for all } j \neq k.
\end{aligned} \tag{10.7}
$$

This last line shows in particular that $N \le C(n)\varepsilon^{2-n}$, where by now C_0 depends only on n.

(ii) As a direct consequence of Theorem 1.7, we can guarantee that

$$
Z_{C_0\varepsilon|\log\varepsilon|} \cap \mathcal{A}_{s,\delta} \subseteq \bigcup_{k=1}^N B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k) \times B^{n-2}_{5C_0\varepsilon}(z_k). \tag{10.8}
$$

(iii) We say that a point $x_k = (y_k, z_k)$ is a *good* point if

$$
\sup_{\varepsilon \le r \le 10C_0\varepsilon} r^{2-n}\mathbf{E}_1(u, \nabla, B^2_1 \times B^{n-2}_r(z_k), \mathbb{R}^{n-2}) \le \eta_0^2,
$$

where with a certain abuse of notation we have set

$$
\begin{aligned}
&\mathbf{E}_1(u, \nabla, B^2_1 \times B^{n-2}_r(z), \mathbb{R}^{n-2}) \\
&:= \int_{B^2_1 \times B^{n-2}_r(z)} \left[\sum_{k=3}^n |\nabla_{e_k} u|^2 + \varepsilon^2 \sum_{(j,k)\neq(1,2)} \omega(e_j, e_k)^2 \right]
\end{aligned} \tag{10.9}
$$

(note the absence of normalization). Let the set of *good indices* be G. We also denote the set of bad ones by $B := \{1, \dots, N\} \setminus G$.

(iv) Again as a direct consequence of Theorem 1.7 we get that

$$
|u_h|, |u| > \frac{3}{4} \quad \text{on} \quad \bigcup_{k=1}^N (B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k) \setminus B^2_{C_0\varepsilon|\log\varepsilon|}(y_k)) \times B^{n-2}_{5C_0\varepsilon}(z_k). \tag{10.10}
$$

Since both u and u_h have degree 1 on each of the previous domains (up to conjugating (u, ∇) on $\mathbb{R}^n$), the *difference of phases* $\theta - \theta_h$ is well-defined on these domains.

(v) We also define a partition of unity $\{\phi_k\}_{k=1}^N$, subordinate to the cylinders:

$$
\phi_k \in C^1_c(B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k) \times B^{n-2}_{5C_0\varepsilon}(z_k)), \quad 0 \le \phi_k \le 1, \tag{10.11}
$$

and

$$
\sum_{k=1}^N \phi_k = 1 \quad \text{on } Z_{C_0\varepsilon|\log\varepsilon|} \cap \mathcal{A}_{s,\delta}.
$$

We also require that $|d\phi_k| \le \tilde{C}\varepsilon^{-1}$ for all $k = 1, \dots, N$.

(vi) Up to modifying the ϕ_k's, we can define $\phi_0 \in C^1_c(\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\log\varepsilon|})$ with $0 \le \phi_0 \le 1$ and

$$\phi_0 = 1 \quad \text{on } \mathcal{A}_{s,\delta} \setminus Z_{5C_0|\varepsilon\log\varepsilon|},$$
$$\phi_0 + \sum_{k=1}^N \phi_k = 1 \quad \text{on } \mathcal{A}_{s,\delta}.$$

Lastly, $|d\phi_0| \le \tilde{C}|\varepsilon\log\varepsilon|^{-1}$.

Note that $C_0 > 0$ is some large enough constant that we are still free to choose later on ($\tilde{C}$ above depends on C_0). In the sequel we also use the following notation for the excess on each ball:

$$\mathbf{E}(k) := \mathbf{E}(u, \nabla, B^2_1 \times B^{n-2}_{10C_0\varepsilon}(z_k), \mathbb{R}^{n-2}).$$

We define $\mathbf{E}(k) = \mathbf{E}_1(k) + \mathbf{E}_2(k)$ similarly. Since the balls $B^{n-2}_{C_0\varepsilon}(z_k)$ are disjoint, the dilated balls have bounded overlap (i.e., at most $C(n)$ of them intersect), giving

$$\sum_{k=1}^N \mathbf{E}(k) \le C \int_{P(\mathcal{A}_{s-\delta,3\delta})} \mathbf{E}_z,$$

and the same holds for $\mathbf{E}_1$ and $\mathbf{E}_2$. With abuse of notation, we also write

$$|\log \mathbf{E}|^2\mathbf{E}(k) := \begin{cases} \int_{B^{n-2}_{5C_0\varepsilon}(z_k)} |\log \mathbf{E}_z|^2 \mathbf{E}_z & \text{if } k \in G, \\ \int_{B^{n-2}_{5C_0\varepsilon}(z_k)} \mathbf{E}_z & \text{if } k \in B. \end{cases} \tag{10.12}$$

Remark 10.3. We will often use the following observation implicitly. For all $z \in B^{n-2}_1$, the results in the previous section show that

$$r^{2-n}\mathbf{E}_1(u, \nabla, B^2_1 \times B^{n-2}_r(z), \mathbb{R}^{n-2}) \le \tilde{\eta}_0^2$$

for an (arbitrarily) small $\tilde{\eta}_0$ and all radii $\Lambda\varepsilon \le r \le \frac{1}{2}$, for some Λ depending on $n, \tilde{\eta}_0$. If $k \in G$ then, for all $z \in B^{n-2}_{5C_0\varepsilon}(z_k)$, we have

$$(C_0\varepsilon)^{2-n}\mathbf{E}_1(u, \nabla, B^2_1 \times B^{n-2}_{C_0\varepsilon}(z), \mathbb{R}^{n-2}) \le 10^{n-2}\eta_0^2.$$

As a consequence of Lemma 5.7, if we take $C_0 \ge C(n)$ large and η_0^2 very small, we have

$$r^{2-n}\mathbf{E}_1(u, \nabla, B^2_1 \times B^{n-2}_r(z), \mathbb{R}^{n-2}) \le \tilde{\eta}_0^2$$

also for $r \in (0, \Lambda\varepsilon)$, since we have C^1 control on the pair at this scale. We now fix $\tilde{\eta}_0$ such that we actually apply Proposition 6.4 for $\eta := \tilde{\eta}_0$, so that *every* $z \in B^{n-2}_{5C_0\varepsilon}(z_k)$ *gives a good slice.*

Gauge fixing on each small cylinder with bounds. Fix $k \in \{1, \dots, n\}$ and consider the unique solution ξ_k to the following Neumann boundary value problem:

$$\begin{cases} \Delta\xi_k = d^*(\alpha - \alpha_h) & \text{in } \mathcal{C}_k, \\ \partial_\nu \xi_k = -(\alpha - \alpha_h)(\nu) & \text{at } \partial\mathcal{C}_k, \\ \int_{\mathcal{A}_k} [(\theta + \xi_k) - \theta_h] = 0, \end{cases} \tag{10.13}$$

where

$$\mathcal{C}_k := B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k) \times B^{n-2}_{5C_0\varepsilon}(z_k)$$

and

$$\mathcal{A}_k := (B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k) \setminus B^2_{C_0\varepsilon|\log\varepsilon|}(y_k)) \times B^{n-2}_{5C_0\varepsilon}(z_k).$$

Recall that $\theta - \theta_h$ is well-defined on $\mathcal{A}_k$. Then we perform the following gauge transformation in $\mathcal{C}_k$: writing $\nabla = d - i\alpha$, we transform

$$(u, \alpha) \mapsto (e^{i\xi_k} u, \alpha + d\xi_k).$$

Since $d^*[(\alpha + d\xi_k) - \alpha_h] = 0$ and $[(\alpha + d\xi_k) - \alpha_h](\nu) = 0$, we can use the Gaffney–Poincaré-type inequality in Lemma B.1:

$$\int_{\mathcal{C}_k} |(\alpha + d\xi_k) - \alpha_h|^2 \le C(n, C_0)|\varepsilon\log\varepsilon|^2 \int_{\mathcal{C}_k} |d\alpha - d\alpha_h|^2. \tag{10.14}$$

We bound separately the contributions of the good set and the bad set, using Lemma 10.4 and Lemma 10.5 below, obtaining

$$\int_{\mathcal{C}_k} |(\alpha + d\xi_k) - \alpha_h|^2 \le C|\log\varepsilon|^2|\log \mathbf{E}|^4 \mathbf{E}(k) + C\varepsilon^{\beta+3n} \tag{10.15}$$

for some $C = C(n, C_0, \beta) = C(n, \beta)$.

Gauge fixing far from the vortex set with bounds. Far from the vortex set, in the set $\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\log\varepsilon|}$, we gauge fix via a function ξ_0 such that $e^{i\xi_0}u/u_h$ is real-valued. Hence, we define

$$\xi_0 := \theta_h - \theta \quad \text{on } \mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\log\varepsilon|}; \tag{10.16}$$

note that a priori ξ_0 is well-defined only in the quotient $\mathbb{R}/2\pi\mathbb{Z}$, but this is enough to have a well-defined gauge transformation (in fact, since the vorticity set is included in a $C\varepsilon$-neighborhood of a graph, we can check that $\theta_h - \theta$ is a well-defined real number).

We can estimate $e^{i\xi_0}u - u_h$ and $(\alpha + d\xi_0) - \alpha_h$ in this domain using the exponential decay (Proposition 4.5), as follows:

$$\begin{aligned} &\int_{\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\log\varepsilon|}} [\varepsilon^{-2}|e^{i\xi_0}u - u_h|^2 + |(\alpha + d\xi_0) - \alpha_h|^2] \\ &\le \int_{\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\log\varepsilon|}} [\varepsilon^{-2}||u| - |u_h||^2 + 2|\alpha - d\theta|^2 + 2|\alpha_h - d\theta_h|^2] \\ &\le C(n)\varepsilon^{-2}e^{-K(n)C_0|\log\varepsilon|}. \end{aligned}$$

In the last inequality, we used the following observation: since each slice intersects the zero set and $|\log\varepsilon| \ge C(n)$, using Lemma 5.7 we see that on

$B^2_1 \times \{z\}$ the distance from $\{|u| \leq 3/4\}$ is comparable with the distance from $(B^2_1 \times \{z\}) \cap \{|u| \leq 3/4\}$.

Taking C_0 large enough, we see that

$$\int_{\mathcal{A}_{s,\delta}\setminus Z_{C_0|\varepsilon\log\varepsilon|}} [\varepsilon^{-2}|e^{i\xi_0}u - u_h|^2 + |(\alpha + d\xi_0) - \alpha_h|^2] \leq \varepsilon^{\beta+3n}. \tag{10.17}$$

Difference of local gauges in the overlap regions. Fix $1 \leq j < k \leq N$ such that

$$\Omega_{j,k} := \mathcal{C}_j \cap \mathcal{C}_k \neq \emptyset.$$

Notice that we can bound the L^2 norm of the difference $d\xi_k - d\xi_j$ as follows:

$$\int_{\Omega_{j,k}} |d\xi_j - d\xi_k|^2 \leq 2\int_{\mathcal{C}_j} |(\alpha + d\xi_j) - \alpha_h|^2 + 2\int_{\mathcal{C}_k} |(\alpha + d\xi_k) - \alpha_h|^2.$$

By (10.15) we then have

$$\int_{\Omega_{j,k}} |d(\xi_j - \xi_k)|^2 \leq C|\log\varepsilon|^4(|\log \mathbf{E}|^2\mathbf{E}(j) + |\log \mathbf{E}|^2\mathbf{E}(k)) + C\varepsilon^{\beta+3n}. \tag{10.18}$$

Our goal is to apply a Poincaré-type inequality on $\Omega_{j,k}$ to estimate $\xi_k - \xi_j$. To this aim, we first look at ξ_j, ξ_k on an appropriate annulus. By the definition of $\mathcal{C}_j, \mathcal{C}_k$ and the structure of the vortex set in Theorem 1.7 we can see that $|x_j - x_k| \leq 20C_0\varepsilon$. We name the midpoint $x_{j,k} = (y_{j,k}, z_{j,k}) := \frac{x_j+x_k}{2}$ and we see that

$$\begin{aligned}\mathcal{A}_{j,k} &:= [B^2_{3C_0\varepsilon|\log\varepsilon|}(y_{j,k}) \setminus B^2_{2C_0\varepsilon|\log\varepsilon|}(y_{j,k})] \times [B^{n-2}_{5C_0\varepsilon}(z_j) \cap B^{n-2}_{5C_0\varepsilon}(z_k)] \\ &\subseteq \mathcal{A}_j \cap \mathcal{A}_k,\end{aligned}$$

which is included in $\Omega_{j,k}$. So we can compute that

$$\int_{\mathcal{A}_{j,k}} |\xi_j - \xi_k|^2 \leq 2\int_{\mathcal{A}_j} |(\theta + \xi_j) - \theta_h|^2 + 2\int_{\mathcal{A}_k} |(\theta + \xi_k) - \theta_h|^2.$$

We know that $(\theta + \xi_j) - \theta_h$ and $(\theta + \xi_k) - \theta_h$ have zero mean on $\mathcal{A}_j$ and $\mathcal{A}_k$, respectively. Hence, we can apply Lemma B.3 on each annulus to see that

$$\int_{\mathcal{A}_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \leq C\int_{\mathcal{A}_j} |d(\theta + \xi_j) - d\theta_h|^2 + C\int_{\mathcal{A}_k} |d(\theta + \xi_k) - d\theta_h|^2,$$

and we can bound

$$|d(\theta + \xi_j) - d\theta_h| \leq |\alpha - d\theta| + |\alpha_h - d\theta_h| + |\alpha + d\xi_j - \alpha_h|.$$

As before, on $\mathcal{A}_j$ we have

$$|\alpha - d\theta|^2 + |\alpha_h - d\theta_h|^2 \leq |u|^{-2}|\nabla u|^2 + |u_h|^{-2}|\nabla_h u_h|^2 \leq \varepsilon^{\beta+3n}$$

by exponential decay, and the same holds for k. Together with (10.15) we thus estimate

$$\int_{\mathcal{A}_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \le C|\log\varepsilon|^4(|\log \mathbf{E}|^2\mathbf{E}(j) + |\log \mathbf{E}|^2\mathbf{E}(k)) + C\varepsilon^{\beta+3n}. \tag{10.19}$$

By (10.18)–(10.19), using Lemma B.3 and Remark B.4, we arrive at

$$\int_{\Omega_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \le C|\log\varepsilon|^4(|\log \mathbf{E}|^2\mathbf{E}(j) + |\log \mathbf{E}|^2\mathbf{E}(k)) + \varepsilon^{\beta+3n}. \tag{10.20}$$

We also need to estimate the difference of $\xi_k - \xi_0$ for all $1 \le k \le N$. Defining

$$\Omega_{0,k} := \mathcal{C}_k \cap [\mathcal{A}_{s,\delta} \setminus Z_{C_0|\varepsilon\log\varepsilon|}],$$

we see that

$$\Omega_{0,k} \subseteq \mathcal{A}'_k := [B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k) \setminus B^2_{(C_0/2)\varepsilon|\log\varepsilon|}(y_k)] \times B^{n-2}_{5C_0\varepsilon}(z_k).$$

Note that by (10.16) we have $\xi_0 = \theta_h - \theta$ in $\Omega_{0,k}$. Hence, we can apply Lemma B.3 and compute that

$$\begin{aligned}
\varepsilon^{-2}\int_{\Omega_{0,k}} |\xi_k - \xi_0|^2 &\le \varepsilon^{-2}\int_{\mathcal{A}'_k} |(\theta + \xi_k) - \theta_h|^2 \\
&\le \int_{\mathcal{A}'_k} |d(\theta + \xi_k) - d\theta_h|^2 \\
&\le \int_{\mathcal{A}'_k} [|\alpha - d\theta|^2 + |\alpha_h - d\theta_h|^2 + |(\alpha + d\xi_k) - \alpha_h|^2],
\end{aligned}$$

where again we used the fact that $(\theta + \xi_k) - \theta_h$ has zero mean on $\mathcal{A}_k \subset \mathcal{A}'_k$. Summing the previous bounds and noting that there are at most $(C\varepsilon^{2-n})^2$ pairs (j,k), while any point belongs to at most $C = C(n, C_0)$ domains $\Omega_{j,k}$, we arrive at

$$\begin{aligned}
&\sum_{0\le j<k\le N} \int_{\Omega_{j,k}} \varepsilon^{-2}|\xi_j - \xi_k|^2 \\
&\le C|\log\varepsilon|^4 \int_{P(\mathcal{A}_{s-\delta,3\delta})} |\log \mathbf{E}_z|^2\mathbf{E}_z + C\mathbf{E} + C\varepsilon^{\beta+1},
\end{aligned} \tag{10.21}$$

for some $C = C(n, \beta)$ (recall (10.12)).

Constructing the global gauge via the partition of unity. Recall the definition of the partition of unity in (10.11). We define the global gauge transformation function as follows:

$$\xi := \sum_{k=0}^{N} \phi_k \xi_k \quad \text{on } \mathcal{A}_{s,\delta}. \tag{10.22}$$

Then we estimate

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} \varepsilon^{-2} \left| e^{i\xi} u - \sum_{k=0}^{N} \phi_k e^{i\xi_k} u \right|^2 \\
&\le \varepsilon^{-2} \int_{\mathcal{A}_{s,\delta}} \sum_{k=0}^{N} \phi_k |\xi - \xi_k|^2 \\
&\le 2 \sum_{j<k} \int_{\Omega_{j,k}} \varepsilon^{-2} |\xi_j - \xi_k|^2 \\
&\le C |\log \varepsilon|^4 \int_{P(\mathcal{A}_{s-\delta,3\delta})} |\log \mathbf{E}_z|^2 \mathbf{E}_z + C\mathbf{E} + C\varepsilon^{\beta+1}.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} \varepsilon^{-2} |e^{i\xi} u - u_h|^2 \\
&\le 2 \int_{\mathcal{A}_{s,\delta}} \left[\varepsilon^{-2} \left| e^{i\xi} u - \sum_{k=0}^{N} \phi_k e^{i\xi_k} u \right|^2 + \sum_{k=0}^{N} \phi_k |e^{i\xi_k} u - u_h|^2 \right] \qquad (10.23) \\
&\le C |\log \varepsilon|^8 \int_{P(\mathcal{A}_{s-\delta,3\delta})} |\log \mathbf{E}_z|^2 \mathbf{E}_z + C\mathbf{E} + C\varepsilon^{\beta+1},
\end{aligned}$$

where we used Lemma 10.4 and Lemma 10.5 to estimate the term involving $e^{i\xi_k} u - u_h$.

Moreover, for the connection part, we can bound

$$\begin{aligned}
&\int_{\mathcal{A}_{s,\delta}} |(\alpha + d\xi) - \alpha_h|^2 \\
&\le 2 \int_{\mathcal{A}_{s,\delta}} \left[\sum_{k=0}^{N} \phi_k |(\alpha + d\xi_k) - \alpha_h|^2 + \left| \sum_{k=0}^{N} d\phi_k \xi_k \right|^2 \right]. \qquad (10.24)
\end{aligned}$$

The first term is bounded by (10.15) and (10.17). We are left to bound the last term. Since the functions ϕ_k form a partition of unity, we have

$$\sum_{k=0}^{N} d\phi_k(z) = 0.$$

We can then write

$$\sum_{k=0}^{N} d\phi_k \xi_k = \sum_{j,k=0}^{N} \phi_j d\phi_k (\xi_k - \xi_j).$$

Since $|d\phi_k| \le C\varepsilon^{-1}$, the last term above is bounded by

$$C\varepsilon^{-2} \sum_{j<k} \int_{\Omega_{j,k}} |\xi_j - \xi_k|^2,$$

which is a quantity that we already estimated. Combining these bounds, we see that

$$\int_{\mathcal{A}_{s,\delta}} [\varepsilon^{-2}|e^{i\xi}u - u_h|^2 + |(\alpha + d\xi) - \alpha_h|^2]$$
$$\leq C|\log\varepsilon|^8 \int_{P(\mathcal{A}_{s-\delta,3\delta})} [\mathbf{E}_z + \mathbf{1}_{\mathcal{G}^\eta}\mathbf{E}_z|\log\mathbf{E}_z|^2] + \varepsilon^\beta.$$

This is indeed the desired conclusion. □

We now turn to the bounds which were postponed in the previous proof.

Lemma 10.4. *Assume that $k \in G$. Then*

$$\int_{\mathcal{C}_k} \varepsilon^{-2}|e^{i\xi_k}u - u_h|^2 \leq C|\log\varepsilon|^8|\log\mathbf{E}|^2\mathbf{E}(k) + C\varepsilon^{\beta+3n}$$

and

$$\int_{\mathcal{C}_k} \varepsilon^2|d\alpha - d\alpha_h|^2 \leq C|\log\varepsilon|^4\eta^{-2}|\log\mathbf{E}|^2\mathbf{E}(k) + C\varepsilon^{\beta+3n}.$$

Proof. We bound each part separately.

Estimating $d\alpha - d\alpha_h$. Recalling the definition of $\mathbf{E}_1$ in (10.9) and the construction of α_h, we have

$$\int_{\mathcal{C}_k} \varepsilon^2|d\alpha - d\alpha_h|^2 \leq \int_{\mathcal{C}_k} \varepsilon^2|d\alpha(e_1,e_2) - d\alpha_h(e_1,e_2)|^2 + C\mathbf{E}_1(k)$$
$$+ C\int_{B^{n-2}_{5C_0\varepsilon}(z_k)} |dh|^2.$$

We are going to use some estimates from [31] which are slightly more refined than (4.14). Compared to the main result of [31], these hold under some additional assumptions, which are however satisfied on good slices: in particular, for any $z \in B^{n-2}_{5C_0\varepsilon}(z_k)$, the function u vanishes linearly at a unique point along the slice $B^2_1 \times \{z\}$. We will often compare u with the function u_{h_0}, where h_0 is the function built in Proposition 6.6, whose graph approximates the zero set; along the good slice, this function vanishes at the same point as u, and is just a translation of the standard degree-one planar solution.

Specifically, using an ε-rescaling of (4.14) and Theorem 4.8 (applied with $N = 1$), we have the following estimate:

(10.25)
$$\int_{B^2_1\times\{z\}} [\varepsilon^{-2}||u| - |u_{h_0}||^2 + |u_{h_0}|^{2+1/2}|(\alpha - d\theta)_{(1,2)} - (\alpha_{h_0} - d\theta_{h_0})_{(1,2)}|^2$$
$$+ \varepsilon^2|d\alpha(e_1,e_2) - d\alpha_{h_0}(e_1,e_2)|^2]$$
$$\leq C\mathbf{E}_z,$$

for an absolute constant C, where the subscript $(1,2)$ means that we restrict the one-form along the slice.

Now by the construction in Proposition 10.1 we can see that (u_h, α_h), along the slice $B_1^2 \times \{z\}$, is equal to (u_{h_0}, α_{h_0}) translated to vanish at the barycenter $\Phi_{\chi(x_1,x_2)}(z)$. As shown in Lemma A.1, the translation is by a vector v with $|v| \le C\varepsilon|\log \mathbf{E}_z|\sqrt{\mathbf{E}_z} + e^{-K/\varepsilon}$. By the mean value theorem, we then have

(10.26)
$$\begin{aligned}\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}} &[\varepsilon^2|d\alpha_h(e_1,e_2) - d\alpha_{h_0}(e_1,e_2)|^2 + |(\alpha_h - \alpha_{h_0})_{(1,2)}|^2 \\ &+ \varepsilon^{-2}|u_h - u_{h_0}|^2] \\ &\le C\varepsilon^2|\log\varepsilon|^2 \cdot |v|^2 \cdot C\varepsilon^{-4} \\ &\le C|\log\varepsilon|^2|\log \mathbf{E}_z|^2\mathbf{E}_z + e^{-K/\varepsilon},\end{aligned}$$

since $\mathbf{E}_z$ is bounded on good slices and the differential of each quantity (such as $\varepsilon d\alpha_h(e_1, e_2)$ and so on) is bounded by $C\varepsilon^{-2}$. The claimed estimate follows by combining the previous bounds (together with item (iv) from Proposition 6.4, which gives $|dh|^2(z) \le C\mathbf{E}_z + e^{-K/\varepsilon}$).

Estimating $e^{i\xi_k}u - u_h$. Writing formally $u = |u|e^{i\theta}$ and using a similar notation for u_h and u_{h_0}, recall that on the annulus

$$\mathcal{A}_{k,z} := [B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k) \setminus B^2_{C_0\varepsilon|\log\varepsilon|}(y_k)] \times \{z\}$$

the differences $\theta - \theta_h$, $\theta - \theta_{h_0}$ and $\theta_h - \theta_{h_0}$ are well-defined. We record the following estimate:

$$\int_{\mathcal{A}_{k,z}} \varepsilon^{-2}|\theta_h - \theta_{h_z}|^2 \le C|\log \mathbf{E}_z|^2\mathbf{E}_z. \tag{10.27}$$

This holds again by the mean value theorem, since $|(d\theta_h)_{(1,2)}|(y) \le C|y - y_k|^{-1}$. We are going to use the Caffarelli–Kohn–Nirenberg-type inequality from Lemma B.2, which implies that

$$\int_{B^2_R(y_k)} |y - h_0(z)|^2|f(y)|^2 \le CR^{3/2}\int_{B^2_R(y_k)} |y - h_0(z)|^{2+1/2}|df(y)|^2,$$

for $f \in C^1_c(B^2_R(y_k))$, with $R := 5C_0\varepsilon|\log\varepsilon|$ (since there exists a biLipschitz transformation sending $B^2_R(y_k)$ to itself and mapping the origin to $h_0(z)$). Recalling that the standard degree-one solution vanishes linearly at the origin, by the construction of u_{h_0} in Proposition 10.1 we have

$$C^{-1} \le \frac{|u_{h_0}|(y,z)}{\min\{\varepsilon^{-1}|y - h_0(z)|, 1\}} \le C$$

on the good slice, for some universal constant C. Moreover,

$$1 \le \frac{\varepsilon^{-1}|y - h_0(z)|}{\min\{\varepsilon^{-1}|y - h_0(z)|, 1\}} \le C|\log\varepsilon| \quad \text{for all } y \in B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k).$$

Hence, given a C^1 function f on $B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}$ vanishing near the boundary, we can write

$$\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}}|u_{h_0}|^2|f|^2\le C\varepsilon^2|\log\varepsilon|^4\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}}|u_{h_0}|^{2+1/2}|df|^2. \tag{10.28}$$

To estimate $ue^{i\xi_k}-u_{h_0}$, we first notice that u and u_{h_0} have the same unique zero point (with the same degree around it), and hence the difference $\theta-\theta_{h_0}$ gives a well-defined smooth function on the full slice.

We define a cut-off $\chi:B^2_1\to\mathbb{R}$ with $\chi=1$ on $B^2_{C_0\varepsilon|\log\varepsilon|}(y_k)$ and $\chi=0$ outside of $B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)$, with $|d\chi|\le C|\varepsilon\log\varepsilon|^{-1}$. Then we use the first term of (10.25) to bound $|u|-|u_{h_z}|$ and (10.28) to see that

$$\begin{aligned}&\varepsilon^{-2}\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}}\chi^2|e^{i\xi_k}u-u_{h_0}|^2\\&\le C\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}}\chi^2\left[\frac{||u|-|u_{h_0}||^2}{\varepsilon^2}+\frac{|u_{h_0}|^2}{\varepsilon^2}|(\theta+\xi_k)-\theta_{h_0}|^2\right]\\&\le C\mathbf{E}_z+C(\mathbf{I}+\mathbf{II}),\end{aligned}$$

where

$$\begin{aligned}\mathbf{I}&:=|\log\varepsilon|^4\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}}|u_{h_0}|^{2+1/2}\chi^2|d(\theta+\xi_k-\theta_{h_0})_{(1,2)}|^2,\\\mathbf{II}&:=|\log\varepsilon|^4\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}}|u_{h_0}|^{2+1/2}|d\chi|^2|\theta+\xi_k-\theta_{h_0}|^2.\end{aligned}$$

First we estimate $\mathbf{I}$ using the second term in (10.25) and (10.26) (to replace α_{h_0} with α_h):

$$\mathbf{I}\le C|\log\varepsilon|^6|\log\mathbf{E}_z|^2\mathbf{E}_z+C|\log\varepsilon|^4\int_{B^2_{5C_0\varepsilon|\log\varepsilon|}(y_k)\times\{z\}}\chi^2|(\alpha+d\xi_k)-\alpha_h|^2.$$

Then we estimate $\mathbf{II}$: we note that $|d\chi|$ is supported in $\mathcal{A}_{k,z}$ and $|d\chi|\le C|\varepsilon\log\varepsilon|^{-1}$, and hence we can use (10.27) to estimate

$$\mathbf{II}\le C|\log\varepsilon|^2|\log\mathbf{E}_z|^2\mathbf{E}_z+C|\log\varepsilon|^2\int_{\mathcal{A}_{k,z}}\varepsilon^{-2}|\theta+\xi_k-\theta_h|^2.$$

Putting **I** and **II** together and integrating over $B^{n-2}_{5C_0\varepsilon}(z_k)$, we see that

$$\begin{aligned}
&\int_{\mathcal{C}_k} \varepsilon^{-2}|e^{i\xi_k}u - u_{h_0}|^2 \\
&\leq C|\log\varepsilon|^6|\log\mathbf{E}|^2\mathbf{E}(k) + C|\log\varepsilon|^4\int_{\mathcal{C}_k}|\alpha + d\xi_k - \alpha_h|^2 \\
&\quad + C|\log\varepsilon|^2\int_{\mathcal{A}_k}\varepsilon^{-2}|\theta + \xi_k - \theta_h|^2 \\
&\leq C|\log\varepsilon|^8|\log\mathbf{E}|^2\mathbf{E}(k) + C|\log\varepsilon|^2\int_{\mathcal{A}_k}\varepsilon^{-2}|\theta + \xi_k - \theta_h|^2 + \varepsilon^{\beta+3n},
\end{aligned}$$

where we used (10.15) (which uses only the previous bound on $d\alpha - d\alpha_h$). Recalling that we imposed

$$\int_{\mathcal{A}_k}(\theta + \xi_k - \theta_h) = 0,$$

we can apply Lemma B.3 (suitably rescaled) and (10.15) another time to see that

$$\begin{aligned}
&|\log\varepsilon|^2\int_{\mathcal{A}_k}\varepsilon^{-2}|\theta + \xi_k - \theta_h|^2 \\
&\leq C|\log\varepsilon|^2\int_{\mathcal{A}_k}|d(\theta + \xi_k) - d\theta_h|^2 \\
&\leq C|\log\varepsilon|^2\int_{\mathcal{A}_k}[|\alpha - d\theta|^2 + |\alpha_h - d\theta_h|^2 + |(\alpha + d\xi_k) - \alpha_h|^2] \\
&\leq C|\log\varepsilon|^6|\log\mathbf{E}|^2\mathbf{E}(k) + \varepsilon^{\beta+3n}
\end{aligned} \tag{10.29}$$

up to taking C_0 large enough (the last inequality follows from the exponential decay of energy); combining these bounds with (10.26), we get the desired bound for $e^{i\xi_k}u - u_h$. □

Lemma 10.5. *For $k \in B$ we have*

$$\int_{\mathcal{C}_k}[\varepsilon^{-2}|e^{i\xi_k}u - u_h|^2 + \varepsilon^2|d\alpha - d\alpha_h|^2] \leq C|\log\varepsilon|^2\mathbf{E}_1(k).$$

Proof. On the bad set we simply use L^∞ bounds: we have

$$\begin{aligned}
&\int_{\mathcal{C}_k}[\varepsilon^{-2}|e^{i\xi_k}u - u_h|^2 + \varepsilon^2|d\alpha - d\alpha_h|^2] \\
&\leq C|\log\varepsilon|^2\varepsilon^{n-2} \\
&\leq C|\log\varepsilon|^2\mathbf{E}_1(k),
\end{aligned} \tag{10.30}$$

where we used the definition of bad index in the last inequality. □

Corollary 10.6. *We have*

$$\int_{\mathcal{A}_{s,\delta}} [\varepsilon^{-2}|e^{i\xi}u - u_h|^2 + |(\alpha + d\xi) - \alpha_h|^2]$$
$$\leq C(n,\beta)|\log\varepsilon|^{10} \int_{P(\mathcal{A}_{s-\delta,3\delta})} \mathbf{E}_z + \varepsilon^\beta.$$

Proof. Recalling that for a good z the sliced excess $\mathbf{E}_z$ is small, it suffices to split the integral of $\mathbf{E}_z|\log\mathbf{E}_z|^2$ on the two sets $\{\mathbf{E}_z \leq \varepsilon^{\beta+1}\}$ and $\{\mathbf{E}_z > \varepsilon^{\beta+1}\}$. On the second one, we bound $|\log\mathbf{E}_z|^2 \leq C|\log\varepsilon|^2$, while on the first one we have $\mathbf{E}_z|\log\mathbf{E}_z|^2 \leq C\varepsilon^{\beta+1}|\log\varepsilon|^2$. □

11. Proof of a stronger decay for local minimizers

11.1. **Strong approximation of the excess for minimizers.** In this section we use variational arguments and the estimates from Proposition 10.2 to construct competitors. As a consequence, we prove that the full excess $\mathbf{E}$ is well approximated by the Dirichlet energy of a harmonic approximation w built as in Proposition 7.2.

Proposition 11.1 (Strong harmonic approximation of minimizers)**.** *For any $\nu,\beta > 0$ and any radius $0 < s < 1$ there exist three small constants $\varepsilon_0(n,s,\nu,\beta), \tau_0(n,s,\nu,\beta), \eta_0(n,\nu,\beta) > 0$ with the following properties. Let (u,∇) be a minimizer of E_ε defined on $B_2^n(0)$, with $\varepsilon \leq \varepsilon_0$ and the energy bound*

$$\frac{1}{|B_2^{n-2}|}\int_{B_2^n} e_\varepsilon(u,\nabla) \leq 2\pi + \tau_0.$$

After a suitable rotation, let $h : B_1^{n-2} \to B_1^2$ be the Lipschitz approximation defined in Proposition 6.4 with $\eta := \eta_0$. Then the following holds, assuming

$$Ce^{-K/\varepsilon} \leq C\varepsilon^\beta \leq \mathbf{E} := \mathbf{E}(u,\nabla,B_2^n,\mathbb{R}^{n-2})$$

for some $C = C(n,\nu,\beta)$ and $K = K(n)$: there exists a harmonic function $w : B_1^{n-2} \to \mathbb{R}^2$ such that

- *(i)* $\int_{B_1^{n-2}} |dw|^2 \leq C(n)$;
- *(ii) we have*

$$\int_{B_1^{n-2}} \left| \frac{h - (h)_{B_1^{n-2}}}{\sqrt{\mathbf{E}}} - w \right|^2 \leq \nu,$$

  where $(h)_{B_1^{n-2}}$ is the average;
- *(iii) most importantly, we have*

$$\int_{B_s^{n-2}} \mathbf{E}_z \leq \mathbf{E}\int_{B_s^{n-2}} \frac{|dw|^2}{2} + \nu\mathbf{E}.$$

Proof. We prove the statement by compactness and contradiction. Fix ν,β,s and assume that there exist sequences $\varepsilon_k,\tau_k \to 0$ and a sequence of minimizers (u_k,∇_k) for E_{ε_k} with the previous energy bound for $\tau_0 = \tau_k$, violating the conclusion. Moreover, let $h_k : B_1^{n-2} \to B_1^2$ be the Lipschitz approximation for the threshold η_0, to be chosen below.

Lower bound on the energy of the given pair. First of all, recalling item (i) in Proposition 6.4 we have

$$\int_{B_1^{n-2}} |dh_k|^2 \le C(n)\mathbf{E}^{(k)}, \quad \mathbf{E}^{(k)} := \mathbf{E}(u_k, \nabla_k, B_2^n, \mathbb{R}^{n-2}),$$

for k large enough. Hence, up to a subsequence, we can extract a weak limit

$$\frac{h_k - (h_k)_{B_1^{n-2}}}{\sqrt{\mathbf{E}^{(k)}}} \rightharpoonup w$$

in $W^{1,2}$, so that

$$\int_{B_1^{n-2}} |dw|^2 \le C(n).$$

By Lemma 7.1 and Proposition 7.2, w is harmonic with $w(0) = 0$. This shows that the first two conclusions hold, so we must have

$$\int_{B_s^{n-2}} \mathbf{E}_z^{(k)} > \mathbf{E}^{(k)} \int_{B_s^{n-2}} \frac{|dw|^2}{2} + \nu \mathbf{E}^{(k)}. \tag{11.1}$$

By Lemma 5.8 and the bound $Ce^{-K/\varepsilon} \le \frac{\nu}{5}\mathbf{E}^{(k)}$, this gives

$$\frac{1}{2\pi}\int_{B_1^2 \times B_s^{n-2}} e_{\varepsilon_k}(u_k, \nabla_k) > |B_s^{n-2}| + \mathbf{E}^{(k)} \int_{B_s^{n-2}} \frac{|dw|^2}{2} + \frac{4\nu}{5}\mathbf{E}^{(k)}.$$

Let $a, b \in (s, 1)$ with $a < b$, which we write as $b = a + 4\delta$. Calling $\mathcal{G}^k$ the good set for (u_k, ∇_k), since the indicator function $\mathbf{1}_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \to 1$ strongly $L^2(B_a^{n-2} \setminus B_s^{n-2})$, we have

$$\mathbf{1}_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \frac{dh_k}{\sqrt{\mathbf{E}^{(k)}}} \rightharpoonup dw$$

weakly in this space, and hence

$$\int_{B_a^{n-2} \setminus B_s^{n-2}} \frac{|dw|^2}{2} \le \liminf_{k \to \infty} \int_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \frac{|dh_k|^2}{2\mathbf{E}^{(k)}}.$$

Using item (iv) in Proposition 6.4 and the assumption $\mathbf{E}^{(k)} \ge C\varepsilon_k^\beta \ge Ce^{-K/\varepsilon_k}$, we deduce

$$\int_{B_a^{n-2} \setminus B_s^{n-2}} \frac{|dw|^2}{2} \le \liminf_{k \to \infty} \int_{(B_a^{n-2} \setminus B_s^{n-2}) \cap \mathcal{G}^k} \frac{\mathbf{E}_z^{(k)}}{\mathbf{E}^{(k)}}.$$

Combined with (11.1), this gives

$$\int_{B_a^{n-2}} \mathbf{E}_z^{(k)} > \mathbf{E}^{(k)} \int_{B_a^{n-2}} \frac{|dw|^2}{2} + \frac{3\nu}{4}\mathbf{E}^{(k)}.$$

Using again Lemma 5.8 and $\mathbf{E}^{(k)} \geq C\varepsilon_k^{\beta} \geq Ce^{-K/\varepsilon_k}$, we obtain

$$\frac{1}{2\pi}\int_{B_1^2\times B_a^{n-2}} e_{\varepsilon_k}(u_k,\nabla_k) > |B_a^{n-2}| + \mathbf{E}^{(k)}\int_{B_a^{n-2}}\frac{|dw|^2}{2} + \frac{3\nu}{5}\mathbf{E}^{(k)}. \tag{11.2}$$

Note that, for a fixed small $\delta > 0$ to be specified later, we can find a and $b = a + 4\delta$ in $(s, 1)$ such that

$$\int_{B_b^{n-2}\setminus B_a^{n-2}}[|dh_k|^2 + \mathbf{E}^{(k)}|dw|^2 + \mathbf{E}_z^{(k)}] \leq C(n,s)\delta\mathbf{E}^{(k)}, \tag{11.3}$$

along a subsequence, by the classical pigeonhole argument.

Now we take a cut-off function χ such that $\chi = 1$ on B_a^{n-2} and $\chi = 0$ outside of $B_{a+\delta}^{n-2}$, and we let

$$f_k := (1-\chi)h_k + \chi(\sqrt{\mathbf{E}^{(k)}}w + (h_k)_{B_1^{n-2}}).$$

Since $\|h_k - (h_k)_{B_1^{n-2}} - \sqrt{\mathbf{E}^{(k)}}w\|_{L^2}^2 = o(\mathbf{E}^{(k)})$, the Dirichlet energy of f_k on $B_b^{n-2}\setminus B_a^{n-2}$ is

$$\int_{B_b^{n-2}\setminus B_a^{n-2}}\left[(1-\chi)^2\frac{|dh_k|^2}{2} + \mathbf{E}^{(k)}(2\chi-\chi^2)\frac{|dw|^2}{2}\right] + o(\mathbf{E}^{(k)}).$$

In particular, by (11.3) we have

$$\int_{B_b^{n-2}\setminus B_a^{n-2}}|df_k|^2 \leq C\delta\mathbf{E}^{(k)}. \tag{11.4}$$

We apply Proposition 10.1 to obtain a new pair (u_{f_k},∇_{f_k}).

Construction of the competitor. We want to glue the latter to (u_k,∇_k) in a suitable annular region and obtain a new pair whose energy in $B_1^2\times B_b^{n-2}$ is strictly lower than (u_k,∇_k), obtaining a contradiction to minimality. From now on, we restrict attention to the region $B_1^2\times B_b^{n-2}$. We will also drop the subscript k in the sequel. Note that $f = h$ on $B_{a+4\delta}^{n-2}\setminus B_{a+\delta}^{n-2}$.

For technical reasons, it will be convenient to glue on an annulus of width $\sqrt{\varepsilon}$. We first select $t \in [a+2\delta, a+3\delta]$ such that

$$\int_{B_{t+2\sqrt{\varepsilon}}^{n-2}\setminus B_{t-\sqrt{\varepsilon}}^{n-2}}[\mathbf{E}_z + |dh|^2] \leq C(n,s,\delta)\sqrt{\varepsilon}\mathbf{E}. \tag{11.5}$$

We first apply Proposition 10.2 and Corollary 10.6 to replace (u,∇) with a gauge-equivalent pair, still denoted (u,∇), such that $\frac{u}{|u|} = \frac{u_f}{|u_f|}$ on $(B_1^2\setminus B_{1/2}^2)\times B_b^{n-2}$, with

$$\int_{\mathcal{A}}[\varepsilon^{-2}|u-u_h|^2 + |\alpha-\alpha_h|^2] \leq C|\log\varepsilon|^{10}\int_{P(\hat{\mathcal{A}})}\mathbf{E}_z + \varepsilon^{\beta},$$

where $\mathcal{A} := B_1^2 \times (B_{t+\sqrt{\varepsilon}}^{n-2} \setminus B_t^{n-2})$ and $\hat{\mathcal{A}} := B_1^2 \times (B_{t+2\sqrt{\varepsilon}}^{n-2} \setminus B_{t-\sqrt{\varepsilon}}^{n-2})$. In particular, by (11.5) we have

(11.6)
$$\int_{\mathcal{A}} [\varepsilon^{-2}|u-u_h|^2 + |\alpha-\alpha_h|^2] \le C\sqrt{\varepsilon}|\log\varepsilon|^{10} \int_{P(\hat{\mathcal{A}})} \mathbf{E}_z + \varepsilon^\beta = o(\mathbf{E}) + \varepsilon^\beta,$$

where the notation $o(\mathbf{E}) = o(\mathbf{E}^{(k)})$ indicates a quantity infinitesimal with respect to $\mathbf{E}^{(k)}$, as $k \to \infty$.

We take another cut-off function φ with $\varphi = 1$ on $B_1^2 \times B_t^{n-2}$ and $\varphi = 0$ outside of $B_1^2 \times B_{t+\sqrt{\varepsilon}}^{n-2}$. On $B_1^2 \times B_b^{n-2}$, we define

$$\tilde{u} := (1-\varphi)u + \varphi u_h, \quad \tilde{\alpha} := (1-\varphi)\alpha + \varphi\alpha_h.$$

We claim that

$$\int_{P(\mathcal{A})} \tilde{\mathbf{E}}_z \le o(\mathbf{E}) + C\varepsilon^\beta. \tag{11.7}$$

Once this is done, using Lemma 5.8, we obtain

$$\frac{1}{2\pi}\int_{P(\mathcal{A})} e_\varepsilon(\tilde{u}, \tilde{\nabla}) \le |P(\mathcal{A})| + o(\mathbf{E}_z) + C\varepsilon^\beta,$$

and hence by Proposition 10.1, together with (11.3) and (11.4), we get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{B_1^2\times B_b^{n-2}} e_\varepsilon(\tilde{u}, \tilde{\nabla}) \\
&\le |B_b^{n-2}| + (1+O(\eta_0^2))\int_{B_b^{n-2}} \frac{|df|^2}{2} + \int_{B_b^{n-2}\setminus B_a^{n-2}} \mathbf{E}_z + o(\mathbf{E}) + C\varepsilon^\beta \\
&\le |B_b^{n-2}| + \mathbf{E}\int_{B_a^{n-2}} \frac{|dw|^2}{2} + \int_{B_b^{n-2}\setminus B_a^{n-2}} [|df|^2 + \mathbf{E}_z] + C\eta_0^2\mathbf{E} + o(\mathbf{E}) + C\varepsilon^\beta \\
&\le |B_b^{n-2}| + \mathbf{E}\int_{B_a^{n-2}} \frac{|dw|^2}{2} + \frac{\nu}{5}\mathbf{E},
\end{aligned}$$

once we take η_0 and δ small enough. In the same way, (11.2) gives

$$\frac{1}{2\pi}\int_{B_1^2\times B_b^{n-2}} e_\varepsilon(u, \nabla) > |B_b^{n-2}| + \mathbf{E}\int_{B_a^{n-2}} \frac{|dw|^2}{2} + \frac{2\nu}{5}\mathbf{E}.$$

This gives a contradiction: near $\partial B_1^2 \times B_{t+\sqrt{\varepsilon}}^{n-2}$, using the fact that $\tilde{u}$ and u_f have the same phase, it is easy to modify $(\tilde{u}, \tilde{\nabla})$ in order to make it agree with (u, ∇) (while this already holds on $\partial B_1^2 \times (B_b^{n-2} \setminus B_{t+\sqrt{\varepsilon}}^{n-2})$), in a way which changes the energy by $O(e^{-K/\varepsilon}) \le \frac{\nu}{5}\mathbf{E}$: it is enough to interpolate between the two pairs on the set

$$(B_1^2 \setminus B_{1/2}^2) \times B_b^{n-2},$$

using the fact that here the energy density is exponentially small, and hence we can write $|\tilde{u} - u_f| = |(1-|\tilde{u}|) - (1-|u_f|)| \le e^{-K/\varepsilon}$ and $|\tilde{\alpha} - \alpha_f| \le e^{-K/\varepsilon}$

(since, writing $\tilde{u} = e^{i\tilde{\theta}}$, we have $|d\tilde{\theta} - \tilde{\alpha}| \le |\tilde{u}|^{-1}|\tilde{\nabla}\tilde{u}|$ and similarly $|d\tilde{\theta} - \alpha_f| = |d\theta_f - \alpha_f| \le |u_f|^{-1}|\nabla_f u_f|$).

Bounding the energy on the interpolation annulus. It remains to check the previous claim. We first write

$$2\pi\tilde{\mathbf{E}}_z = \int_{B_1^2\times\{z\}} \sum_{j\ge 3}|\tilde{\nabla}_{e_j}\tilde{u}|^2 + \sum_{(j,j')\ne(1,2)} \varepsilon^2|d\tilde{\alpha}(e_j,e_{j'})|^2 + |\tilde{\nabla}_{e_1}\tilde{u} + i\tilde{\nabla}_{e_2}\tilde{u}|^2$$
$$+ \left|\varepsilon d\tilde{\alpha}(e_1,e_2) - \frac{1-|\tilde{u}|^2}{2\varepsilon}\right|^2$$
$$=: \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV}.$$

We start by bounding $\mathbf{II}$: we have

$$d\tilde{\alpha} = (1-\varphi)d\alpha + \varphi d\alpha_h + d\varphi \wedge (\alpha_h - \alpha).$$

Hence, using the fact that $|d\varphi| \le C\varepsilon^{-1/2}$, we have

$$\mathbf{II} \le C\mathbf{E}_z + C|dh|^2(z) + C\varepsilon^2 \cdot C\varepsilon^{-1}\int_{B_1^2\times\{z\}} |\alpha_h - \alpha|^2.$$

The last term is bounded by the left-hand side of (11.6); together with (11.5), this gives the desired bound.

As for $\mathbf{IV}$, we note that

$$\frac{1-|\tilde{u}|^2}{2\varepsilon} = (1-\varphi)\frac{1-|u|^2}{2\varepsilon} + \varphi\frac{1-|u_h|^2}{2\varepsilon} + O(\varepsilon^{-1}|u-u_h|),$$

and hence $\mathbf{IV}$ is the squared norm of

$$(1-\varphi)\left[\varepsilon d\alpha(e_1,e_2) - \frac{1-|u|^2}{2\varepsilon}\right] + \varphi\left[\varepsilon d\alpha_h(e_1,e_2) - \frac{1-|u_h|^2}{2\varepsilon}\right]$$
$$+ O(\sqrt{\varepsilon}|\alpha-\alpha_h|) + O(\varepsilon^{-1}|u-u_h|).$$

Thus, we have

$$\mathbf{IV} \le C\mathbf{E}_z + C\varepsilon\int_{B_1^2\times\{z\}}|\alpha-\alpha_h|^2 + C\varepsilon^{-2}\int_{B_1^2\times\{z\}}|u-u_h|^2,$$

again bounded by (11.5) and (11.6).

We finally turn to $\mathbf{I}$; the bound for $\mathbf{III}$ is obtained in the same way and hence will be skipped. We note that

$$\tilde{\nabla}\tilde{u} = d[(1-\varphi)u + \varphi u_h] - i[(1-\varphi)u + \varphi u_h][(1-\varphi)\alpha + \varphi\alpha_h]$$
$$= (1-\varphi)\nabla u + \varphi\nabla_h u_h + (u_h - u)d\varphi + O(|u-u_h||\alpha-\alpha_h|).$$

Hence,

$$\int_{B_1^2\times\{z\}}|\tilde{\nabla}_{e_j}\tilde{u}|^2 \le C\mathbf{E}_z + C|dh|^2(z) + C\varepsilon^{-1}\int_{B_1^2\times\{z\}}|u-u_h|^2$$
$$+ C\int_{B_1^2\times\{z\}}|\alpha-\alpha_h|^2.$$

Again, the last two terms are bounded by the left-hand side of (11.6). This completes the proof of (11.7), and hence the proof of the proposition. □

11.2. **Proof of Theorem 1.10.** In this section we finish the proof of the stronger decay of excess for minimizers. We rescale $B^n_1(0)$ to $B^n_2(0)$ and apply Proposition 11.1, with some $s \in (0, \frac{1}{2})$ and $\nu > 0$ to be chosen later and with $\beta + 1$ in place of β. We obtain that either $\mathbf{E} = \mathbf{E}(u, \nabla, B^n_2(0), \mathbb{R}^{n-2}) \le C\varepsilon^{\beta+1}$ or the conclusions of Proposition 11.1 hold true (provided that the picture is rotated in such a way that $\mathbf{E}$ is small enough).

In the first situation, we clearly have $\min_S \mathbf{E}(u, \nabla, B^n_2, S) \le \varepsilon^\beta$ for ε small enough and we are done. Hence, in the sequel, we can assume that we are in the second situation.

We will assume for simplicity that $\mathbb{R}^{n-2}$ minimizes $\mathbf{E}(u, \nabla, B^n_2, \cdot)$ and that

$$|dw(0)| \le \delta \tag{11.8}$$

with $\delta > 0$ to be chosen momentarily.

Since $|dw(z) - dw(0)| \le s \sup_{B^{n-2}_s} |D^2 w|$ on B^{n-2}_s, we have

$$\int_{B^{n-2}_s} |dw|^2 \le C(n)s^{n-2}\delta^2 + C(n)s^n \sup_{B^{n-2}_s} |D^2 w|^2 \le C(n)s^{n-2}(\delta^2 + s^2),$$

where the last inequality comes from the bound $\|dw\|_{L^2} \le C(n)$ and standard elliptic estimates.

By item (iii) from Proposition 11.1 we then have

$$\int_{B^{n-2}_s} \mathbf{E}_z \le \mathbf{E} \int_{B^{n-2}_s} \frac{|dw|^2}{2} + \nu\mathbf{E} \le C(n)s^{n-2}(\delta^2 + s^2 + s^{2-n}\nu)\mathbf{E}.$$

This immediately gives

$$\mathbf{E}(u, \nabla, B^n_s, \mathbb{R}^{n-2}) \le C(n)(\delta^2 + s^2 + s^{2-n}\nu)\mathbf{E}.$$

The theorem follows under the assumption (11.8) by taking δ, s and *subsequently* ν small enough. The general case can be reduced to this one by the very same argument of Section 8.2; the only differences here are that we use item (ii) from Proposition 11.1 in order to bound

$$\|h - (h)_{B^{n-2}_1} - \sqrt{\mathbf{E}} w\|^2_{L^2} \le \nu\mathbf{E}$$

and that $\mathbf{E}_1$ is replaced by $\mathbf{E}$ throughout that argument.

11.3. **Proof of Theorem 1.9: the case of minimizers.** We prove the following theorem, which contains the second part of Theorem 1.9.

Theorem 11.2. *For $n \ge 2$, there exists $\tau_0(n) > 0$ such that the following holds. If (u, ∇) is an entire, local minimizer for the energy E_1, with $u(0) = 0$ and the energy bound*

$$\frac{1}{|B^{n-2}_R|} \int_{B^n_R} \left[|\nabla u|^2 + |F_\nabla|^2 + \frac{1}{4}(1 - |u|^2)^2 \right] \le 2\pi + \tau_0,$$

then (u,∇) is two-dimensional. More precisely, we have $(u,\nabla)=P^*(u_0,\nabla_0)$ up to a change of gauge, where P is the orthogonal projection onto a two-dimensional subspace and (u_0,∇_0) is the standard degree-one solution of Taubes [51] (or its conjugate), centered at the origin.

Proof. We can assume $n\geq 3$. We proceed exactly as in the proof of Theorem 1.9: taking any $\beta>n-2$, it is enough to prove that

$$\limsup_{R\to\infty} R^\beta \min_S \mathbf{E}_1(u,\nabla,B^n_R,S)<\infty,$$

which in turn implies that

$$\lim_{R\to\infty} R^{n-2}\min_S \mathbf{E}_1(u,\nabla,B^n_R,S)=0.$$

This follows from the stronger excess decay statement for minimizers, using the same iteration argument employed in the proof of Theorem 1.9. □

Appendix A. Barycenter and variance of good slices

We show two lemmas which give a more refined control of a critical pair on a *good slice* $B^2_1\times\{z\}$, with $z\in\mathcal{G}^\eta$, the good set defined in (6.3). We assume that (u,∇) is a critical pair for E_ε, defined on $B^2_1\times B^{n-2}_1$, with $\varepsilon\leq\varepsilon_0$ and

$$E_\varepsilon(u,\nabla)\leq|B^{n-2}_1|(2\pi+\tau_0)$$

(as well as (4.1)–(4.2)). Under this assumption, we have

$$\int_{(B^2_{3/4}\setminus B^2_{1/2})\times\{z\}} e_\varepsilon(u,\nabla)\leq e^{-K(n)/\varepsilon}$$

for $z\in B^{n-2}_{3/4}$, since this part of the slice is far from the vorticity set. Recall that the barycenter

$$h(z)=\Phi_{\chi(x_1,x_2)}(z)$$

was defined using a cut-off function χ supported in $B^2_{3/4}$, with $\chi=1$ on $B^2_{1/2}$ (the notation in the subscript means $\chi\cdot(x_1,x_2)$).

Lemma A.1 (Barycenter of a good slice)**.** *For $\varepsilon_0,\tau_0,\eta_0>0$ small enough, if $\eta\leq\eta_0$ and $z\in\mathcal{G}^\eta$, then we have the following estimate (for a possibly different $K=K(n)$):*

$$|h(z)-h_0(z)|\leq C(n)\varepsilon|\log(\mathbf{E}_2)_z|(\mathbf{E}_2)_z^{1/2}+e^{-K/\varepsilon},$$

where h_0 is the map from Proposition 6.6 giving the zero set on good slices.

In other words, the barycenter of the good slice is close to the actual zero of u here (unique in $B^2_{1/2}\times\{z\}$).

Proof. Recall that, by definition, we have

$$h(z)=\Phi_{\chi(x_1,x_2)}(z)=\frac{1}{2\pi}\int_{B^2_{1/2}\times\{z\}}(x_1,x_2)J(u,\nabla)(e_1,e_2)+Ce^{-K/\varepsilon}.$$

Since the integral of the Jacobian on $B^2_{1/2} \times \{z\}$ is $2\pi + O(e^{-K/\varepsilon})$ (see, e.g., the proof of [42, Lemma 6.11]), we get

$$h(z) - h_0(z) = \frac{1}{2\pi} \int_{B^2_{1/2} \times \{z\}} [(x_1, x_2) - h_0(z)] J(u, \nabla)(e_1, e_2) + Ce^{-K/\varepsilon}.$$

On the other hand, using the notation from Proposition 10.1, we have

$$\left| \frac{1}{2\pi} \int_{B^2_{1/2} \times \{z\}} [(x_1, x_2) - h_0(z)] J(u_{h_0}, \nabla_{h_0})(e_1, e_2) \right| \leq Ce^{-K/\varepsilon},$$

by symmetry of the standard planar solution.

Moreover, $u(\cdot, z)$ vanishes linearly at $h_0(z)$, as observed in Lemma 5.7. We can then apply a rescaling of (4.15) in Theorem 4.8, which gives

(A.1)
$$\int_{B^2_{1/2} \times \{z\}} |J(u, \nabla)(e_1, e_2) - J(u_{h_0}, \nabla_{h_0})(e_1, e_2)| \leq C\sqrt{(\mathbf{E}_2)_z} + Ce^{-K/\varepsilon}.$$

Selecting a radius $C(n)\varepsilon \leq r \leq \frac{1}{4}$, we have

$$e_\varepsilon(u, \nabla)(y, z) \leq C(n)\varepsilon^{-2} e^{-K|y - h_0(z)|/\varepsilon} \quad \text{on } [B^2_{1/2} \setminus B^2_r(h_0(z))] \times \{z\}$$

for a possibly different K, since as observed in Lemma 5.7 the distance from the vorticity set Z is comparable to the distance from

$$Z \cap (B^2_{3/4} \times \{z\}) \subseteq B^2_{C(n)\varepsilon}(h_0(z)) \times \{z\},$$

on good slices. Hence,

$$\begin{aligned} &\int_{B^2_{1/2} \times \{z\}} |(x_1, x_2) - h_0(z)| |J(u, \nabla)(e_1, e_2) - J(u_{h_0}, \nabla_{h_0})(e_1, e_2)| \\ &\leq r \int_{B^2_r(h_0(z)) \times \{z\}} |J(u, \nabla)(e_1, e_2) - J(u_{h_0}, \nabla_{h_0})(e_1, e_2)| \\ &\quad + C\varepsilon^{-2} \int_{B^2_{1/2} \setminus B^2_r(h_0(z))} |y - h_0(z)| e^{-K|y - h_0(z)|/\varepsilon}\, dy \\ &\leq Cr\sqrt{\mathbf{E}_z} + C\varepsilon e^{-Kr/\varepsilon} \end{aligned}$$

for a possibly different K. Taking $r := M\varepsilon |\log \mathbf{E}_z|$ for big enough M, we get

$$r\sqrt{\mathbf{E}_z} + \varepsilon e^{-Kr/\varepsilon} \leq M\varepsilon\sqrt{\mathbf{E}_z} |\log \mathbf{E}_z| + \varepsilon\sqrt{\mathbf{E}_z} \leq C(n)\varepsilon\sqrt{\mathbf{E}_z} |\log \mathbf{E}_z|$$

(recall that $\mathbf{E}_z \leq \frac{1}{2}$, by definition of good set), unless $r < C(n)\varepsilon$ or $r > \frac{1}{4}$. The situation $r < C(n)\varepsilon$ cannot happen, once M is taken large enough, while in the last case we obtain $\mathbf{E}_z \leq e^{-K'/\varepsilon}$ and thus we are done again, by taking $r := \frac{1}{4}$ above. □

Next we show that on a good slice the variance is close to $\varepsilon^2 v_0$, where v_0 is the variance of the standard degree-one planar solution.

Lemma A.2 (Variance of a good slice). *For any $\sigma \in (\varepsilon, 1)$ such that $|h_0(z)| \le \sigma$, we have the following estimate on good slices, for any $c \in \mathbb{R}^2$ with $|c| \le \sigma$:*

$$\left| \frac{1}{2\pi} \int_{B^2_{1/2} \times \{z\}} [(x_1 - c_1)^2 + (x_2 - c_2)^2] e_\varepsilon(u, \nabla) - \varepsilon^2 v_0 \right|$$
$$\le C(n)\varepsilon^2 |\log(\mathbf{E}_2)_z|^2 \sqrt{(\mathbf{E}_2)_z} + C(n)\sigma^2 (\mathbf{E}_1)_z + C(n)|h(z) - c|^2 \sqrt{(\mathbf{E}_2)_z}$$
$$+ C(n) e^{-K\sigma/\varepsilon},$$

for a possibly different $K = K(n)$.

Proof. First of all, since the integrand in the definition of excess $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ upper bounds $e_\varepsilon(u, \nabla) - J(u, \nabla)$, we can replace $e_\varepsilon(u, \nabla)$ with $J(u, \nabla)$, up to an error bounded as follows: for $\mathbf{E}_1$, we bound separately the contribution of $B^2_{2\sigma}$ and the complement (where we use exponential decay) obtaining the error

$$9\sigma^2 (\mathbf{E}_1)_z + C(n) e^{-K\sigma/\varepsilon};$$

as for $\mathbf{E}_2$, we argue as in the previous proof, obtaining the error

$$C(n)|h_0(z) - c|^2 (\mathbf{E}_2)_z + C(n)\varepsilon^2 |\log(\mathbf{E}_2)_z|^2 (\mathbf{E}_2)_z + e^{-K/\varepsilon},$$

where the first term comes from replacing c with the actual location $h_0(z)$ of the zero. Moreover, by definition of v_0, we have

$$\frac{1}{2\pi} \int_{B^2_{1/2} \times \{z\}} |(x_1, x_2) - h_0(z)|^2 J(u_{h_0}, \nabla_{h_0})(e_1, e_2) = \varepsilon^2 v_0 + O(e^{-K/\varepsilon});$$

since

$$|(x_1, x_2) - h_0(z)|^2 - |(x_1, x_2) - c|^2 = 2\langle (x_1, x_2) - h_0(z), c - h_0(z) \rangle$$
$$- |c - h_0(z)|^2$$

and

$$\int_{B^2_{1/2} \times \{z\}} [(x_1, x_2) - h_0(z)] J(u_{h_0}, \nabla_{h_0}) = O(e^{-K/\varepsilon}),$$

we obtain

$$\frac{1}{2\pi} \int_{B^2_{1/2} \times \{z\}} |(x_1, x_2) - c|^2 J(u_{h_0}, \nabla_{h_0})(e_1, e_2) = |h_0(z) - c|^2 + \varepsilon^2 v_0 + O(e^{-K/\varepsilon}).$$

As in the previous proof, we can replace $J(u_0, \nabla_0)$ with $J(u, \nabla)$ here, up to an error of the form $C(n)(|h_0(z) - c|^2 + \varepsilon^2 |\log \mathbf{E}_z|^2)\sqrt{(\mathbf{E}_2)_z}$ (using (4.15) from Theorem 4.8). Finally, we can bound

$$|h_0(z) - c|^2 \le 2|h_0(z) - h(z)|^2 + 2|h(z) - c|^2$$
$$\le 2|h(z) - c|^2 + C(n)\varepsilon^2 |\log(\mathbf{E}_2)_z|^2 (\mathbf{E}_2)_z$$

using the previous proposition, and the claim follows. ∎

Remark A.3. Since $t \mapsto t|\log t|^2$ is concave for $t > 0$ small enough, we have

$$\int_S \mathbf{E}_z |\log \mathbf{E}_z|^2 \le \left(\int_S \mathbf{E}_z\right) \left|\log\left(\fint_S \mathbf{E}_z\right)\right|^2 \le C \left(\fint_S \mathbf{E}_z\right) \left|\log\left(\fint_S \mathbf{E}_z\right)\right|^2$$

for sets $S \subseteq \mathcal{G}^\eta$ of measure comparable with 1.

Appendix B. Poincaré–Gaffney-type inequalities

In the construction of the interpolation gauge in Proposition 10.2 we make frequent use of Poincaré-type inequalities for functions and differential forms. These inequalities are well known; we present the special cases used in this paper for the convenience of the reader.

The following lemma is a consequence of results first appeared in the original paper of Gaffney [27] (see also [35] for a systematic treatment on manifolds with boundary), but for our application we need it to hold uniformly for cylinders of the form $B_1^2 \times B_r^{n-2}$ of arbitrarily small width $r > 0$.

Lemma B.1 (Poincaré–Gaffney-type inequality for thin cylinders)**.** *Given a* 1*-form* $\alpha \in \Omega^1(\overline{B}_1^2 \times \overline{B}_r^{n-2})$ *with* $r \le 1$ *and the Neumann boundary condition* $\alpha(\nu) = 0$ *on* $\partial(B_1^2 \times B_r^{n-2})$*, the following inequality holds:*

$$\int_{B_1^2 \times B_r^{n-2}} |\alpha|^2 \le C(n) \int_{B_1^2 \times B_r^{n-2}} [|d\alpha|^2 + |d^*\alpha|^2].$$

Proof. Since $B_1^2 \times B_r^{n-2}$ is a convex domain and $\iota_\nu \alpha = 0$ at its boundary, we can apply [15, Remark 9] to see that

$$\int_{B_1^2 \times B_r^{n-2}} |\nabla\alpha|^2 \le \int_{B_1^2 \times B_r^{n-2}} [|d\alpha|^2 + |d^*\alpha|^2].$$

Now we rescale the domain with the map $\phi : B_1^2 \times B_1^{n-2} \to B_1^2 \times B_r^{n-2}$ given by

$$\phi(x_1, \dots, x_n) := (x_1, x_2, rx_3, \dots, rx_n),$$

and define $\tilde\alpha(x) := \alpha(\phi(x))$ (notice that this is different from the pullback $\phi^*(\alpha)$). Then we claim that there exists a constant $C(n) > 0$ such that

$$\int_{B_1^2 \times B_1^{n-2}} |\tilde\alpha|^2 \le C(n) \int_{B_1^2 \times B_1^{n-2}} |\nabla\tilde\alpha|^2 . \tag{B.1}$$

We prove this by compactness and contradiction. By homogeneity, suppose there exists a sequence $\tilde\alpha_k$ with $\iota_\nu \tilde\alpha_k = 0$ on $\partial(B_1^2 \times B_1^{n-2})$ and

$$\int_{B_1^2 \times B_1^{n-2}} |\tilde\alpha_k|^2 = 1, \quad \lim_{k\to\infty} \int_{B_1^2 \times B_1^{n-2}} |\nabla\tilde\alpha_k|^2 = 0.$$

Note that by the display above we have the bound $\|\tilde\alpha_k\|_{W^{1,2}(B_1^2 \times B_1^{n-2})} \le 2$ for all large $k \ge 0$. Up to extracting a subsequence, we can assume that $\tilde\alpha_k$

converges weakly to $\tilde{\alpha}_\infty$ in $W^{1,2}$. By Rellich–Kondrachov, the convergence is strong in L^2. Thus,

$$\int_{B_1^2\times B_1^{n-2}} |\tilde{\alpha}_\infty|^2 = 1, \quad \nabla\tilde{\alpha}_\infty = 0.$$

Hence, $\tilde{\alpha}_\infty = v$ is a constant covector. The boundary condition passes to the limit, giving that $v(\nu) = 0$ on $\partial(B_1^2 \times B_1^{n-2})$; since the normal vectors to the boundary of this domain span all of $\mathbb{R}^n$, we get that $v = 0$, a contradiction establishing (B.1). Then we compute that

$$\begin{aligned}\int_{B_1^2\times B_r^{n-2}} |\alpha|^2 &= r^{n-2}\int_{B_1^2\times B_1^{n-2}} |\tilde{\alpha}|^2 \\ &\le C(n)r^{n-2}\int_{B_1^2\times B_1^{n-2}} |\nabla\tilde{\alpha}|^2 \\ &\le C(n)\int_{B_1^2\times B_r^{n-2}} |\nabla\alpha|^2 \\ &\le C(n)\int_{B_1^2\times B_r^{n-2}} [|d\alpha|^2 + |d^*\alpha|^2],\end{aligned}$$

as desired. □

The next lemma is a weighted Poincaré estimate for functions in two dimensions.

Lemma B.2. *There exists a constant $C > 0$ such that for any compactly supported function $f \in C_c^1(B_R^2)$ the following weighted Poincare type estimate holds:*

$$\int_{B_R^2} |x|^2|f|^2(x) \le CR^{3/2}\int_{B_R^2} |x|^{5/2}|df|^2(x).$$

Proof. By scaling the domain, we can assume that $R = 1$. Then by [13, eq. (1.4)] (for the choice of constants $\alpha := 5/4$, $a := 1$, $p = q = r := 2$, $\gamma, \sigma := 1/4$) we can see that

$$\int_{B_1^2} |x|^2|f|^2(x) \le \int_{B_1^2} |x|^{1/2}|f|^2(x) \le C\int_{B_1^2} |x|^{5/2}|df|^2(x).$$

This is indeed the desired conclusion. □

Lemma B.3 (Poincaré inequality on a thin annulus)**.** *Given $a > b \ge c \ge 0$, there exists a constant $C(n, a, b, c) > 0$ with the following property. Let f be a function in $W^{1,2}((B_a^2 \setminus B_c^2) \times \Omega)$, where $\Omega \subseteq \mathbb{R}^{n-2}$ is a convex bounded domain, such that*

$$\int_{(B_a^2\setminus B_b^2)\times\Omega} f = 0.$$

Then the following Poincaré inequality holds:

$$\int_{(B_a^2 \setminus B_c^2)\times\Omega} |f|^2 \leq C(n,a,b,c)\,\mathrm{diam}(\Omega)^2 \int_{(B_a^2 \setminus B_c^2)\times\Omega} |df|^2.$$

Proof. First we apply the standard Poincaré inequality on each two dimensional slice $(B_a^2 \setminus B_c^2) \times \{z\}$ for any $z \in \Omega$:

$$\int_\Omega \left[\int_{(B_a^2 \setminus B_c^2)\times\{z\}} |f|^2\right] dz$$

$$\leq C(a,b,c) \int_{(B_a^2 \setminus B_c^2)\times\Omega} |df|^2 + \int_\Omega \left|\int_{(B_c^2 \setminus B_b^2)\times\{z\}} f\right|^2 dz.$$

Notice that the function $g(z) := \int_{(B_a^2 \setminus B_b^2)\times\{z\}} f$ has zero average on Ω. Hence we can apply the Poincaré inequality on Ω to see that

$$\int_\Omega |g|^2 \leq C(n)\,\mathrm{diam}(\Omega)^2 \int_\Omega |dg|^2.$$

Indeed, it is well known that the Poincaré inequality on a convex domain holds with a constant depending only on its diameter and n. This yields the desired conclusion. □

Remark B.4. The same conclusion holds if we assume that

$$\int_{(B_a^2 \setminus B_b^2)\times\Omega'} f = 0$$

for some Ω' with $|\Omega'| \geq \alpha|\Omega|$ (the constant depending also on α).

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