Easy ex.: $\dim \mathbb{H} = 2$, $\mathbb{H} \cong \mathbb{C}^2$, ONB: $\{|0\rangle, |1\rangle\}$,
$|0\rangle \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$.

$|\psi\rangle = a|0\rangle + b|1\rangle$, $a, b \in \mathbb{C} \rightsquigarrow \langle\psi| = \bar{a}\langle 0| + \bar{b}\langle 1|$.

What about operators $|\psi\rangle\langle\varphi|$ with $|\psi\rangle, |\varphi\rangle \in \mathbb{H}$?

$|0\rangle\langle 0| \to \begin{pmatrix}1\\0\end{pmatrix}(1\ \ 0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ (projection on first coordinate)

$|1\rangle\langle 1| \to \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$ (" " second " ")

$|0\rangle\langle 1| \longrightarrow \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ $\quad |1\rangle\langle 0| \longrightarrow \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$

$|\varphi\rangle = c|0\rangle + d|1\rangle \xRightarrow{\text{by linearity}} |\varphi\rangle\langle\psi| = \begin{pmatrix}a\bar{c} & a\bar{d}\\ b\bar{c} & b\bar{d}\end{pmatrix}$.

Let $A: \mathbb{H} \to \mathbb{H}$ be a linear operator. We will assume that A is bounded, that is, $\underbrace{\sup\{\|A\psi\| \mid |\psi\rangle \in \mathbb{H}, \|\psi\| = 1\}}_{\|A\|,\ \text{operator norm}} < +\infty$.

The adjoint of A is $A^*: \mathbb{H} \to \mathbb{H}$ s.t.
$\langle A^*\psi|\varphi\rangle = \langle\psi|A\varphi\rangle \quad \forall\, |\psi\rangle, |\varphi\rangle \in \mathbb{H}$ (A^* exists because of Riesz).

A is self-adjoint if $A = A^*$.

Properties:
(1) $(A^*)^* = A$
(2) $c \in \mathbb{C}$, $(cA)^* = \bar{c}A^*$
(3) $A^*_{jk} = \bar{A}_{kj}$
(4) $\langle A\psi| = \langle\psi|A^*$
(5) $(|\psi\rangle\langle\varphi|)^* = |\varphi\rangle\langle\psi|$

$U: \mathbb{H} \to \mathbb{H}$ is a unitary operator if
$\langle U\psi|U\varphi\rangle = \langle\psi|\varphi\rangle \quad \forall\, |\psi\rangle, |\varphi\rangle \in \mathbb{H}$.

Equivalent characterizations:
- $U^*U = \mathrm{Id}$
- $\|U\psi\| = \|\psi\| \ \forall\, |\psi\rangle \in \mathbb{H}$ (ex.; hint: parallelogram identity)

Remark: $\{e_j\}_{j\in I}$ ONB for $\mathbb{H}$, $\dim \mathbb{H} < +\infty$; then $\{\tilde{e}_j\}_{j\in I}$ with $\tilde{e}_j = Ue_j$ is again an ONB.

A operator on $\mathbb{H}$, $|\psi\rangle \in \mathbb{H}$, $|\psi\rangle \neq 0$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$ if $A|\psi\rangle = \lambda|\psi\rangle$.

Spectrum of A: $\sigma(A) = \{\lambda \in \mathbb{C} \mid (A - \lambda\mathrm{Id})$ is not invertible$\}$.
λ eigenvalue $\Rightarrow \lambda \in \sigma(A)$. In finite dim., $\sigma(A) =$ eigenvalues of A.

Let $A = A^*$, λ eigenvalue of A, $|\psi\rangle$ corresponding eigenvector.
$\langle\psi|A\psi\rangle = \langle\psi|\lambda\psi\rangle = \lambda\langle\psi|\psi\rangle = \lambda\|\psi\|^2$
$\quad \|$
$\langle A\psi|\psi\rangle = \langle\lambda\psi|\psi\rangle = \bar{\lambda}\langle\psi|\psi\rangle = \bar{\lambda}\|\psi\|^2 \Rightarrow \lambda \in \mathbb{R}$.

Let U be unitary, eigenvalue of U.
$\|\psi\| = \|U\psi\| = \|\lambda\psi\| = |\lambda|\,\|\psi\| \Rightarrow |\lambda| = 1$.

Let A be a compact operator on $\mathbb{H}$, $A = A^*$. (compact: always verified in finite dim.)
Then A can be diagonalized wrt an ONB of $\mathbb{H}$.
Let $\{\lambda_j\}_{j\in I}$ be the eigenvalues of A, then $A = \sum_{j,\alpha} \lambda_j |e_{j,\alpha}\rangle\langle e_{j,\alpha}|$,
where λ_j has mult. d_j and $e_{j,\alpha}$, $\alpha = 1, \dots, d_j$ are the elements of the ONB that span the eigenspace of λ_j (for $\lambda_j \neq 0$).

Projection operators: $P: \mathbb{H} \to \mathbb{H}$ is a projection if $P^2 = P$.
Let $\mathbb{K}$ be a subspace of $\mathbb{H}$; P is a projection onto $\mathbb{K}$ if $\forall\, |\varphi\rangle \in \mathbb{H}$, $P|\varphi\rangle \in \mathbb{K}$.
For instance: let $|\psi\rangle \in \mathbb{H}$, $\|\psi\| = 1$; projection operator on $\mathrm{Span}\{|\psi\rangle\}$: $P = |\psi\rangle\langle\psi|$.
P is an orthogonal projection if $P^2 = P$ and $P = P^*$.
In this case there exists an orthonormal set $\{|\psi_j\rangle\}_{j\in J}$ s.t. $P = \sum_j |\psi_j\rangle\langle\psi_j|$.

$A, B: \mathbb{H} \to \mathbb{H}$, the commutator of A and B is
$[A, B] = AB - BA$. A and B commute $\iff [A, B] = 0$.

Def.: $A: \mathbb{H} \to \mathbb{H}$ is positive if $\langle\psi|A\psi\rangle \geq 0 \ \forall\, |\psi\rangle \in \mathbb{H}$,
is strictly positive if $\langle\psi|A\psi\rangle = 0 \iff |\psi\rangle = 0$.

Trace operator: $A: \mathbb{H} \to \mathbb{H}$, $\{e_j\}$ ONB,
it is the linear map $A \to \sum_j \langle e_j|Ae_j\rangle = \mathrm{tr}(A)$.
Also $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.