Primitive Cohomology

(X,ω) cpt Kähler of (comp.) dim. n.

it's what goes to 0 after wedging with ω enough (but not too many) times

Idea: rep. theory of $\mathfrak{sl}_2$: $N=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $[H,N]=-2N$,

$N(\alpha)=\alpha\wedge\omega$, $H(\alpha)=\dim_{\mathbb{C}}X-\deg\alpha$.

Hodge-Riemann bilinear relations

(X,ω) cpt Kähler of dim. d, $H_0^k(X,\mathbb{R})$. Then

$Q(\alpha,\beta)=\underbrace{\pm}_{\substack{\text{universal}\\ \text{in } (k,d)}}\int_X \alpha\wedge\beta\wedge\omega^{d-k}$ has the property that

$\langle\alpha,\beta\rangle=Q(C\alpha,\bar\beta)$ is a positive def. hermitian inner product where $C_{|H^{p,q}}=i^{p-q}$.

Period domain

Data: $H_{\mathbb{Z}}=\underbrace{\overbrace{H_0^k(X,\mathbb{Q})}^{\text{primitive cohomology}}\cap H^k(X,\mathbb{Z})}_{\text{lattice}}$, $Q: H_{\mathbb{Z}}\otimes H_{\mathbb{Z}}\to\mathbb{Z}$, weight k,

$h^{p,q}$ Hodge numbers, $\overline{h^{p,q}}=h^{q,p}$, $\sum h^{p,q}=\operatorname{rk}H_{\mathbb{Z}}$.

$D=\{F^\bullet H_{\mathbb{C}}$ decreasing filtration $|$ $F^\bullet H_{\mathbb{C}}$ is a pure Hodge structure of weight k with $\operatorname{rk}(F^p\cap\overline{F^{k-p}})=h^{p,q}$ which is polarized by Q, i.e. it satisfies the H.-R. bilinear relation $\langle\alpha,\beta\rangle=Q(C\alpha,\bar\beta)$ is hermitian and makes the Hodge decomposition orthogonal$\}$.

$f: X\to S$ holo. map of comp. man. which is proper and of max rank and X is Kähler then we get a period map, which is given locally by $\varphi(s)=H_0^k(X_s,\mathbb{C})$, $X_s=f^{-1}(s)$, $\varphi: S\to D/\Gamma$. To have an integral structure, one really needs X projective.

Ex.: $h^{1,0}=h^{0,1}=g$, $D=\{g\times g$ matrices with positive def. imaginary part$\}$.

Ex.: $h^{2,0}=h^{0,2}=1$, $h^{1,1}=h$. H.-R. bilinear relations $\Rightarrow Q(F^2,F^2)=0$,

$\langle H^{2,0},H^{0,2}\rangle=0$, $\alpha,\beta\in H^{2,0}$, $Q(C\alpha,(\bar\beta))=0$, $Q(C\alpha,\beta)=0\Rightarrow Q(\alpha,\beta)=0$.

$h^{2,0}=1\Rightarrow F^2=\mathbb{C}\alpha$; $0<\langle\alpha,\bar\alpha\rangle=Q(C\alpha,\bar\alpha)=-Q(\alpha,\bar\alpha)$. So

$D=\{\mathbb{C}\alpha \mid \alpha\in H_{\mathbb{C}}$ with $Q(\alpha,\alpha)=0$, $Q(\alpha,\bar\alpha)<0\}$.

$(H^{2,0}\oplus H^{0,2})^{\perp}=H^{1,1}$, so $H^{2,0}$ determines all $H^{p,q}$s

Q is non-degenerate and $(-1)^{\text{weight}}$-sym.

$G_{\mathbb{R}}=\operatorname{Aut}_{\mathbb{R}}(Q)$ acts transitively on D.

$Q(F^p,F^{n-p+1})=0$, $G_{\mathbb{C}}\curvearrowright\check{D}=\{F^\bullet H_{\mathbb{C}} \mid \dim_{\mathbb{C}}F^p=\sum_{q\geq p}h^{q,k-q}$ and satisfies $Q(F^p,F^{n-p+1})=0\}$.

compact dual

acts transitively