Teo.: se f e g sono unif. lim. e lipschitz, allora $\exists$ proc. $(X_t^x)_{x\in\mathbb{R}^d,\, t\geq 0}$, $\mathbb{P}$-q.c. cont. in (t,x) e t.c. $\forall x\in\mathbb{R}^d$ $(X_t^x)_{t\geq 0}$ risolve $e_x(f,g)$.

Dim.: siano $X=X^x$, $Y=X^y$ sol. di $e_x(f,g)$, $e_y(f,g)$ (rispetto allo stesso BM).

$$\mathbb{E}\Big[\sup_{t\le T}|X_t-Y_t|^{\underset{\text{ci serve } p \text{ grande}}{p}}\Big]=\mathbb{E}\Big[\sup_{t\le T}\Big|x-y+\int_0^t (g(s,X)-g(s,Y))\,ds+\int_0^t (f(s,X)-f(s,Y))\,dB_s\Big|^p\Big]\le$$

$$\le c(p)\Big(|x-y|^p+\mathbb{E}\Big[\sup_{t\le T}\Big|\int_0^t (g(s,X)-g(s,Y))\,ds\Big|^p+\Big|\int_0^t (f(s,X)-f(s,Y))\,dB_s\Big|^p\Big]\Big).$$

$$\mathbb{E}\Big[\sup_{t\le T}\Big|\int_0^t (g(s,X)-g(s,Y))\,ds\Big|^p\Big]\le T^{p-1}\,\mathbb{E}\Big[\int_0^T |g(s,X)-g(s,Y)|^p\,ds\Big]\le$$
$$\le T^{p-1}K^p\int_0^T \mathbb{E}\Big[\sup_{0\le r\le s}|X_r-Y_r|^p\Big]ds.$$

$$\mathbb{E}\Big[\sup_{t\le T}\Big|\int_0^t (f(s,X)-f(s,Y))\,dB_s\Big|^p\Big]\overset{\text{BDG, arrestando a } T}{\le}$$
$$\le c(p)\,\mathbb{E}\Big[\Big(\int_0^T |f(s,X)-f(s,Y)|^2\,ds\Big)^{p/2}\Big]\overset{\text{come sopra}}{\le}$$
$$\le c(p)\,T^{p/2-1}K^p\int_0^T \mathbb{E}\Big[\sup_{0\le r\le s}|X_r-Y_r|^p\Big]ds.$$ Allora

$$\Phi_t:=\mathbb{E}\Big[\sup_{s\le t}|X_s-Y_s|^p\Big]\le c(p)|x-y|^p+c(p)T^{p-1}K^p\int_0^t\Phi_s\,ds+\tilde c(p)T^{p/2-1}K^p\int_0^t\Phi_s\,ds,$$ T fissato;

inoltre, f, g lim. $\Rightarrow \Phi_t<+\infty$. Gronwall $\xRightarrow{c,\,c' \text{ le costanti giuste}}$

$\Rightarrow \Phi_t\le c|x-y|^p e^{c't}\Rightarrow \mathbb{E}\Big[\sup_{0\le t\le T}|X_t^x-X_t^y|^p\Big]\le c(p,K,T)|x-y|^p.$

Per continuità di Kolmogorov, $\exists$ modificazione cont. (di $X^\cdot:\Omega\to C([0,1];\mathbb{R}^d)^\circ$) cont. (in x e t) purché $p>d$.

Ex.: se $x_n\to x$ e $X^{x_n}\xrightarrow{L^p}X$ unif. in t, allora X soddisfa $e_x(f,g)$. $\square$

Dato $(X_t^x)_{x\in\mathbb{R}^d,\, t\ge 0}$, definiamo $\forall x\in\mathbb{R}^d$ $\forall t\ge 0$ $\forall h\in C_0(\mathbb{R}^d)$
$$P_t h(x):=\mathbb{E}[h(X_t^x)].$$

Oss.: $(t,x)\mapsto P_t h(x)$ è cont., lim. e lineare; inoltre,
$$|\mathbb{E}[h(X_t^x)]|\le \mathbb{E}[|h(X_t^x)|\mathbb{1}_{\{|X_t^x|>R\}}]+\mathbb{E}[|h(X_t^x)|\mathbb{1}_{\{|X_t^x|\le R\}}]\le$$
$$\le \sup_{|z|>R}|h(z)|+\sup_{|z|\le R}|h(z)|\,\mathbb{P}(|X_t^x|\le R).$$

Se $|x|>2R$, $\{|X_t^x|\le R\}\subseteq\Big\{\Big|\int_0^t g(s,X^x)\,ds+\int_0^t f(s,X^x)\,dB_s\Big|>R\Big\}\Rightarrow$
$$\Rightarrow \mathbb{P}(|X_t^x|\le R)\le \mathbb{P}\Big(\Big|\int_0^t g(s,X^x)\,ds\Big|>R/2\Big)+\mathbb{P}\Big(\Big|\int_0^t f(s,X^x)\,dB_s\Big|>R/2\Big).$$

$\mathbb{P}\Big(\Big|\int_0^t g(s,X^x)\,ds\Big|>R/2\Big)=0$ se $R/2>t\sup_{0\le s\le t} g(s,X^x)$.

$\mathbb{P}\Big(\Big|\int_0^t f(s,X^x)\,dB_s\Big|>R/2\Big)\overset{\text{Chebychev}}{\le}\mathbb{E}\Big[\Big|\int_0^t f(s,X^x)\,dB_s\Big|^2\Big]/(R/2)^2.$

$$\mathbb{E}\Big[\Big|\int_0^t f(s,X^x)\,dB_s\Big|^2\Big]=\sum_{i=1}^d \mathbb{E}\Big[\Big|\sum_{j=1}^k\int_0^t f_j^i(s,X^x)\,dB_s^j\Big|^2\Big]=$$
$$=\sum_{i=1}^d \mathbb{E}\Big[\Big\langle\sum_{j=1}^k f_j^i\cdot B^j\Big\rangle_t\Big]=\sum_{i=1}^d\sum_{j=1}^k \mathbb{E}\big[((f_j^i)^2\cdot\langle B^j\rangle)_t\big]\le C(f,t,d,k).$$

Quindi $|\mathbb{E}[h(X_t^x)]|\le \sup_{|z|>R}|h(z)|+\|h\|_{C_0}\frac{C}{R^2}$. Per h, t fissati,
$$\lim_{|x|\to+\infty}|P_t h(x)|=0 \quad (\Rightarrow P_t: C_0(\mathbb{R}^d)\to C_0(\mathbb{R}^d)).$$

Domanda: vale Chapman-Kolmogorov? $P_{t+s}h(x)=P_t(P_s h)(x)$?

Se $f(s,\omega)=\tilde f(\omega_s)$ e $g(s,\omega)=\tilde g(\omega_s)$ sì.

Oss.: YW $\Rightarrow$ unicità in legge $\Rightarrow$ se Z è v.a. a valori in $\mathbb{R}^d$ indi. da B allora, dato X^x, il proc. $(X_t^Z)_{t\ge 0}$ risolve $e(f,g)$ con dato iniziale Z.

Oss.: data X^x, il proc. $(X_{t+s}^x)_{t\ge 0}$ risolve $e(f,g)$ con BM $(B_{t+s}-B_s)_{t\ge 0}$ e dato iniziale X_s^x (conti, attenzione a $f\cdot B$) $\Rightarrow$
$\Rightarrow$ la legge di $(X_{t+s}^x)_{t\ge 0}$ è la stessa di $(X_t^Z)_{t\ge 0}$, dove Z è v.a. indi da B con legge uguale a $X_s^x\Rightarrow$
$$\Rightarrow P_{t+s}h(x)=\mathbb{E}[h(X_{t+s}^x)]=\mathbb{E}[h(X_t^Z)]=$$
$$=\mathbb{E}\big[\mathbb{E}[h(X_t^Z)\,|\,Z]\big]\underset{\substack{\mathbb{E}[h(X_t^Z)|Z=z]=\\=\mathbb{E}[h(X_t^z)]=P_t h(z)}}{=}\mathbb{E}[P_t h(Z)]=\mathbb{E}[(P_t h)(X_s^x)]=P_s(P_t h)(x).$$