χ primitivo reale modulo q

$|\gamma_\chi| \geq \frac{\delta}{\mathcal{L}}$

$$\frac{4}{\sigma-\beta_\chi} < \frac{3}{\sigma-1} + \mathrm{Re}\left(\frac{1}{\sigma-1+2i\gamma_\chi}\right) + c\mathcal{L} = \frac{3}{\sigma-1} + \frac{\sigma-1}{(\sigma-1)^2+4\gamma_\chi^2} + c\mathcal{L}$$

$$\sigma = 1+\frac{\delta}{\mathcal{L}} \Rightarrow \frac{4}{\delta/\mathcal{L}+(1-\beta_\chi)} < \frac{3\mathcal{L}}{\delta} + \frac{\mathcal{L}}{5\delta} + c\mathcal{L} \Rightarrow$$

$$\Rightarrow \frac{\mathcal{L}}{\delta+(1-\beta_\chi)\mathcal{L}} < \frac{3\mathcal{L}}{\delta} + \frac{\mathcal{L}(1+5c\delta)}{5\delta} = \frac{16+5c\delta}{5\delta} \Rightarrow$$

$$\Rightarrow \frac{\delta+(1-\beta_\chi)\mathcal{L}}{4} > \frac{20}{16+5c\delta}\delta \Rightarrow$$

$$\Rightarrow (1-\beta_\chi)\mathcal{L} > \frac{20}{16+5c\delta}\delta - \delta = \frac{4\delta-5c\delta^2}{16+5c\delta} \Rightarrow$$

$$\Rightarrow 1-\beta_\chi > \frac{4-5c\delta}{16+5c\delta}\cdot\frac{\delta}{\mathcal{L}} \overset{\delta \text{ piccolo}}{\Longrightarrow}$$

$$\Rightarrow \beta_\chi < 1 - \frac{c_0\delta}{\mathcal{L}}.$$

$$-\frac{1}{\sigma-1} < c\log q - \frac{2(\sigma-\beta_\chi)}{(\sigma-\beta_\chi)^2+\gamma_\chi^2}, \quad \beta_\chi \pm i\gamma_\chi,\ |\gamma_\chi| < \frac{\delta}{\log q}$$

$$\sigma = 1+\frac{2\delta}{\log q} \Rightarrow |\gamma_\chi| < \frac{\sigma-1}{2} \leq \frac{\sigma-\beta_\chi}{2}$$

$$< c\log q - \frac{8}{5(\sigma-\beta_\chi)}$$

$$\frac{8}{5(\sigma-\beta_\chi)} < c\log q + \frac{1}{\sigma-1} = \left(c+\frac{1}{2\delta}\right)\log q \Rightarrow$$

$$\Rightarrow \frac{8\log q}{10\delta+5(1-\beta_\chi)\log q} < \left(\frac{1}{2\delta}+c\right)\log q = \frac{1+2\delta c}{2\delta}$$

$$\frac{10\delta+5(1-\beta_\chi)\log q}{8} > \frac{16\delta}{1+2\delta c} \Rightarrow$$

$$\Rightarrow 5(1-\beta_\chi)\log q > \frac{6\delta-20\delta^2c}{1+2\delta c} \Rightarrow$$

$$\Rightarrow 1-\beta_\chi > \frac{6-20\delta c}{5+10\delta c}\cdot\frac{\delta}{\log q} \overset{\delta \text{ piccolo}}{\Longrightarrow}$$

$$\Rightarrow \beta_\chi < 1 - \frac{c_1\delta}{\log q}.$$ Con $\gamma_\chi = 0$ è simile ma più semplice.

Prop.: $\exists\, c_0 > 0$ t.c. se χ è primitivo modulo q e $\beta_\chi + i\gamma_\chi$ è uno zero di $L(s,\chi)$, allora

$$\beta_\chi < \begin{cases} 1-\frac{c_0}{\log(q|\gamma_\chi|)} & \text{se } |\gamma_\chi| \geq 1 \\ 1-\frac{c_0}{\log q} & \text{se } |\gamma_\chi| \leq 1 \end{cases}$$

e se χ è reale modulo q, esiste al più un solo zero β_0 reale semplice t.c. $L(\beta_0,\chi)=0$ e β_0 non soddisfa la disuguaglianza.

Prop. (Landau): siano χ_1 modulo q_1 e χ_2 modulo q_2 reali primitivi $\chi_1 \neq \chi_2$ e siano β_1, β_2 t.c. $L(\beta_1,\chi_1)=0$, $L(\beta_2,\chi_2)=0$. Allora $\min\{\beta_1,\beta_2\} < 1-\frac{c_1}{\log(q_1q_2)}$ con $c_1 > 0$.

Cor.: poiché si può scegliere $q_1=q_2=q$, allora $\min\{\beta_1,\beta_2\} < 1-\frac{c_1}{2\log q} = 1-\frac{c_2}{\log q}$.

Cor.: $q_1 < q_2 < \dots$, χ_j modulo q_j reale primitivo, $L(\beta_j,\chi_j)=0$ con $\beta_j > 1-\frac{c_3}{\log q_j}$. Allora $q_{j+1} > q_j^2$. $(c_3 = c_1/3)$.

Dim.: (del secondo cor.): per assurdo $q_{j+1} \leq q_j^2$. Allora

$$\min\{\beta_j, \beta_{j+1}\} < 1-\frac{c_1}{\log(q_jq_{j+1})} \leq 1-\frac{c_1}{3\log q_j} < 1-\frac{c_3}{\log q_{j+1}}. \quad \square$$

Cor. (lemma di Page): $\exists\, c_4 > 0$ t.c. $\exists!\, q \leq z$ e $\exists!\, \chi$ è reale primitivo modulo q con $L(\beta_\chi,\chi)=0$ con $\beta_\chi > 1-\frac{c_4}{\log z}$.

Dim.: per assurdo (β_1,χ_1,q_1) e (β_2,χ_2,q_2) con $q_1,q_2 \leq z$ t.c. $\beta_j > 1-\frac{c_4}{\log z}$. $q_1\cdot q_2 \leq z^2 \overset{\text{Landau}}{\Longrightarrow}$

$$\Rightarrow \min\{\beta_1,\beta_2\} < 1-\frac{c_1}{2\log z}. \quad c_4 = \frac{c_1}{2} \Rightarrow \text{assurdo}. \quad \square$$

Prop.: sia χ reale primitivo modulo q. Vale la disuguaglianza

$$L(1,\chi) > \frac{c_0}{\sqrt{q}}.$$

Oss.: $L(\beta_0,\chi)=0 \Rightarrow L(1,\chi) = L(1,\chi)-L(\beta_0,\chi) =$

$$= L'(\sigma_0,\chi)(1-\beta_0) \ll (1-\beta_0)\log^2 q \Rightarrow 1-\beta_0 \geq \frac{c_1}{\sqrt{q}\log^2 q}.$$

$\beta_0 \leq \sigma_0 \leq 1$

σ_0 può stare qui (WLOG) $\left[1-\frac{1}{\log q}, 1\right]$

(χ modulo q qualunque)

Lemma: sia $1-(\log q)^{-1} \leq \sigma \leq 1$. Si ha $L'(\sigma,\chi) \ll \log^2 q$.

Dim.: $\sigma > 1 \Rightarrow L'(\sigma,\chi) = -\sum_{n=1}^{+\infty}\frac{\chi(n)\log n}{n^\sigma} =$

$$= \sigma\int_2^{+\infty}\left(\sum_{n\leq u}\chi(u)\right)\frac{1+\sigma\log u}{u^{\sigma+1}}\,du \quad \text{che conv. unif. per } \sigma \geq \varepsilon.$$

(Sommazione per parti)

$$\left|\sum_{n=1}^{q}\frac{\chi(n)\log n}{n^\sigma}\right| \leq \frac{1}{e}\sum_{n=1}^{q}\frac{\log n}{n} \ll \log^2 q$$

$$n^\sigma \geq \exp\left(\left(1-\frac{1}{\log q}\right)\log n\right) \geq n/e.$$

L'altro pezzo lo maggioro con $\sigma\int_q^{+\infty}\left|\sum_{n\leq u}\chi(u)\right|\frac{1+\log u}{u^2}$

$\hookrightarrow u^{\sigma-1} \geq q^{\sigma-1} \geq q^{-\frac{1}{\log q}} = 1/e$. Allora ho

$$q\int_q^{+\infty}\frac{\log u}{u^2}\,du \ll q\cdot\frac{\log q}{q} = \log q. \quad \square$$

Formule esplicite

Def.: si pone $\psi(x,\chi) = \sum_{n\leq x}\chi(n)\Lambda(n)$ per χ modulo q.

Oss.: se $\chi = \chi_0$, allora $\psi(x,\chi_0) = \sum_{\substack{n\leq x,\\(n,q)=1}}\Lambda(n) = \psi(x) - \sum_{\substack{n\leq x,\\(n,q)>1}}\Lambda(n) =$

$$= \psi(x) + O\left(\sum_{p|q}\sum_{a\leq\frac{\log x}{\log p}}\log p\right) = \psi(x) + O\left(\log x\sum_{p|q}1\right) = \psi(x) + O(\log x\log q) =$$

$$= \psi(x) + O(\log^2(qx)).$$

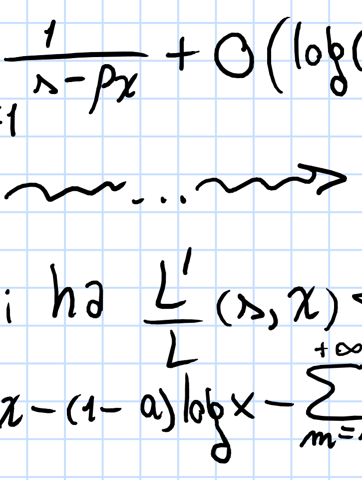
Poniamo $\psi_0(x,\chi) = \sum_{n\leq x}\chi(n)\Lambda(n) + \begin{cases}\frac{1}{2}\chi(x)\Lambda(x) & \text{se } x\in\mathbb{N}\\ 0 & \text{se } x\notin\mathbb{N}\end{cases}$.

$$\chi \neq \chi_0 \Rightarrow \psi_0(x,\chi) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{L'}{L}(s,\chi)\frac{x^s}{s}\,ds + O\left(\frac{x\log^2 x}{T} + \log x\min\left\{1,\frac{x}{T\langle x\rangle}\right\}\right).$$

$c = 1+\frac{1}{\log x}$

1) se $\chi(-1) = -1$ $(a=1)$ $\Rightarrow L(-2m+1,\chi) = 0$ $m\in\mathbb{N}$.
Ci sono: $-\frac{x^{\rho_\chi}}{\rho_\chi}$, $-\frac{L'}{L}(0,\chi)$, $\frac{x^{-2m+1}}{2m-1}$.

2) se $\chi(-1) = 1$ $(a=0)$. Ci sono:
$-\frac{x^{\rho_\chi}}{\rho_\chi}$, $-\log x - \lim_{s\to 0}\left(\frac{L'}{L}(s,\chi)-\frac{1}{s}\right)$, $\frac{x^{-2m}}{2m}$.

Di nuovo il rettangolo 

Stavolta abbiamo

$$\frac{L'}{L}(s,\chi) = \sum_{|\gamma_\chi - T|\leq 1}\frac{1}{s-\rho_\chi} + O(\log(qT)).$$ Imponendo

$|\gamma_\chi \pm T| \gg \frac{1}{\log(qT)}$ $\rightsquigarrow \dots \rightsquigarrow$ $\frac{L'}{L}(s,\chi) \ll \log^2(qT)$.

Per gli altri pezzi si ha $\frac{L'}{L}(s,\chi) \ll \log(qT)$. Si ottiene

$$\psi_0(x,\chi) = -\sum_{|\gamma_\chi|<T}\frac{x^{\rho_\chi}}{\rho_\chi} - b_\chi - (1-a)\log x - \sum_{m=1}^{+\infty}\frac{x^{-2m+a}}{a-2m} + O\left(\frac{x\log^2(qTx)}{T} + \log x\min\left\{1,\frac{x}{T\langle x\rangle}\right\}\right).$$

$$\frac{L'}{L}(s,\chi) = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + \sum_{\rho_\chi}\left(\frac{1}{s-\rho_\chi}+\frac{1}{\rho_\chi}\right) + A_\chi.$$